ПРИКЛАДНА МАТЕМАТИКА

DOI: https://doi.org/10.20535/kpisn.2025.4.343114
UDC 519.832.3

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LINEAR-ACCURACY ONE-BULLET SILENT DUEL WITH PROGRESSING-BY-ONE-THIRD SHOOTING MOMENTS

Background. A finite zero-sum game is considered, which models competitive interaction between two subjects. The subject, referred to as the duelist, must take an action (or, metaphorically, shoot the single bullet) during a standardized time span, where the bullet can be shot at only specified time moments. The duelist benefits from shooting as late as possible, but only when the duelist shoots first.

Objective. The objective is to determine optimal behavior of the duelists for a pattern of the duel discrete progression, by which the tension builds up as the duel end approaches and there are more possibilities to shoot.

Methods. Both the duelists act within the same conditions, and so the one-bullet silent duel is symmetric. Therefore, its optimal value is 0 and the duelists have the same optimal strategies. The shooting accuracy is linear being determined by an accuracy proportionality factor.

Results. Depending on the factor, all pure strategy solutions are found for such duels, whose possible-shooting moments comprise a progression pattern. According to this pattern, every next possible-shooting moment is obtained by adding the third of the remaining span to the current moment. The solutions for this pattern are compared to the known solutions for the geometrical-progression pattern and the pattern whose possible-shooting moments progress in a smoother manner.

Conclusions. The proved assertions contribute another specificity of the progressing-by-one-third shooting moments in linear-accuracy one-bullet silent duels to the games of timing. Compared to duels for other duel discrete progression patterns, the specificity consists in that the duel with progressing-by-one-third shooting moments has a constant interval of lower (weaker) shooting accuracies, at which the duelist possesses an optimal pure strategy. This interval is $\left[\frac{4}{5}; \frac{6}{5}\right]$ that symmetrically breaks the low-accuracy interval $(0; 2)$.

Keywords: one-bullet silent duel; linear accuracy; matrix game; pure strategy solution; progressing-by-one-third shooting moments.

1. Introduction

A one-bullet silent duel is a timing zero-sum game, in which it is unknown to the player (also referred to as the duelist) whether and when the other duelist has fired its bullet until the end of the duel time span [1, 2]. The span is usually interval $[0; 2]$. The bullet is a metaphor for an option to make a decision or take an action [3, 4]. In fact, shooting (or firing) a bullet means making a decision or taking an action during interaction between the two duelists (decision-makers, consumers, entrepreneurs, users, etc.) [5, 6]. The duelist may not fire the bullet until the very last (final) moment to shoot, but then it is nonetheless fired at the final moment, because the action must be taken anyway [2, 7, 8]. The duelist is also featured with an accuracy function which is a nondecreasing function of time [1, 9, 10].

To more realistically simulate interaction between the two duelists, discrete silent duels are

Пропозиція для цитування цієї статті: В.В. Романюк, “Безшумна дуель з однією кулею лінійної влучності та прогресуючими на одну третину моментами пострілу”, *Наукові вісті КПІ*, № 4, с. 7–19, 2025. doi: 10.20535/kpisn.2025.4.343114

Offer a citation for this article: V.Vadim Romanuke, “Linear-accuracy one-bullet silent duel with progressing-by-one-third shooting moments”, *KPI Science News*, No. 4, pp. 7–19, 2025. doi: 10.20535/kpisn.2025.4.343114

considered, in which the duelist can shoot only at specified time moments [1, 3, 4, 11, 12]. The number of such possible shooting moments is finite. The moments of the duel beginning and duel end are included in this number [7, 13, 14]. So, in a discrete duel with possible shooting moments the players' pure strategy sets are

$$X_N = \{x_i\}_{i=1}^{N} = Y_N = \{y_j\}_{j=1}^{N} = T_N = \{t_q\}_{q=1}^{N} \subset [0;1] \quad (1)$$

by

$$t_q < t_{q+1} \quad \forall\, q = \overline{1, N-1} \text{ and } t_1 = 0,\ t_N = 1$$

for $N \in \mathbb{N} \setminus \{1\}$.

It is presumed that both the duelists act within the same conditions, and so the one-bullet silent duel is symmetric. Therefore, its optimal value is 0 and the duelists have the same optimal strategies, although they still can be non-symmetric [3, 11, 13, 15]. The duelist benefits from shooting as late as possible, but only when the duelist shoots first [2, 16, 17]. This is modeled, in particular, by a skew-symmetric payoff matrix [1, 7, 18]

$$\mathbf{K}_N = \left[k_{ij}\right]_{N\times N} = \left[-k_{ji}\right]_{N\times N} = -\mathbf{K}_N^{\mathrm{T}} \quad (2)$$

whose entries

$$k_{ij} = ax_i - ay_j + a^2 x_i y_j \operatorname{sign}\left(y_j - x_i\right) \quad (3)$$

for

$$i = \overline{1, N} \text{ and } j = \overline{1, N} \text{ by } a > 0.$$

The accuracy proportionality factor a defines the duelists' linear accuracy functions [7, 16, 19]

$$p_X(x) = ax, \quad p_Y(y) = ay, \quad (4)$$

through which entry k_{ij} can be generally given as

$$k_{ij} = p_X(x_i) - p_Y(y_j) + $$
$$+ p_X(x_i)\, p_Y(y_j) \operatorname{sign}\left(y_j - x_i\right). \quad (5)$$

Hence, the global objective is to find pure strategy solutions of linear-accuracy one-bullet silent duel (LA1BSD)

$$\langle X_N, Y_N, \mathbf{K}_N \rangle = \left\langle \{x_i\}_{i=1}^{N}, \{y_j\}_{j=1}^{N}, \mathbf{K}_N \right\rangle \quad (6)$$

by (1)–(3).

LA1BSD (6) is called progressive if the density of the duelist's pure strategies between $t_1 = 0$ and $t_N = 1$ progressively grows (in accordance with a definite pattern) as the duelist approaches to the duel end $t_N = 1$ [1, 7, 9, 10, 12, 13]. The duel's shooting-moment progression is quite natural because the tension builds up as the duel end approaches, and thus the duelist must have more possibilities to shoot [6, 11, 20, 21]. A particular interest of applying LA1BSDs exists in advertising, where competitiveness and waiting to attract and harvest more audience data are modeled [22, 23].

2. Known results

The first particular case of the progressive LA1BSD was considered in [15], where

$$t_q = \sum_{l=1}^{q-1} 2^{-l} = \frac{2^{q-1}-1}{2^{q-1}} \quad (7)$$

for $q = \overline{2, N-1}$ and pure strategy solutions had been obtained for any $a \geq 1$, and specific conditions had been found for $a \in (0;1)$ such, at which the duel has a pure strategy solution. Thus, situation

$$\{x_2, y_2\} = \left\{\frac{1}{2}, \frac{1}{2}\right\} \quad (8)$$

is single optimal in duel (6) by (1)–(3), (7), and $a > 1$ for $N \in \mathbb{N} \setminus \{1, 2\}$. Situation (8) is non-optimal by $a \in (0;1)$. However, situation (8) remains single optimal by $a = 1$ for $N \in \mathbb{N} \setminus \{1, 2, 3\}$. The duel by $a = 1$ for $N = 3$ has four optimal situations (8),

$$\{x_3, y_3\} = \{1, 1\}, \quad (9)$$

$$\{x_3, y_2\} = \left\{1, \frac{1}{2}\right\}, \quad (10)$$

$$\{x_2, y_3\} = \left\{\frac{1}{2}, 1\right\}. \quad (11)$$

Situation

$$\{x_2, y_2\} = \{1, 1\} \quad (12)$$

is single optimal by any $a > 0$ in the most trivial case, when $N = 2$ (and thus the duelist can shoot only either at the duel beginning or duel end, which annuls the progressiveness). Situation (9) is the single solution to 3×3 duels by $a \in (0;1)$. For the general case of $N = 2$ article [15] proves that only one $n \in \left\{\overline{3, N-1}\right\}$ exists such that situation

$$\{x_n, y_n\} = \left\{\frac{2^{n-1}-1}{2^{n-1}}, \frac{2^{n-1}-1}{2^{n-1}}\right\} \quad (13)$$

is optimal by

$$a \in \left[\frac{1}{2^{n-1}-1}; \frac{2^{n-2}}{\left(2^{n-1}-1\right)\cdot\left(2^{n-2}-1\right)}\right] \subset (0;1) \quad (14)$$

and situation

$$\{x_N, y_N\} = \{1, 1\} \tag{15}$$

is optimal by

$$a \in \left(0; \frac{1}{2^{N-2}-1}\right] \subset (0; 1) \tag{16}$$

for $N \in \mathbb{N} \setminus \{1, 2, 3\}$. If

$$a = \frac{1}{2^{N-2}-1} \tag{17}$$

then situations (15),

$$\{x_{N-1}, y_{N-1}\} = \left\{\frac{2^{N-2}-1}{2^{N-2}}, \frac{2^{N-2}-1}{2^{N-2}}\right\}, \tag{18}$$

$$\{x_{N-1}, y_N\} = \left\{\frac{2^{N-2}-1}{2^{N-2}}, 1\right\}, \tag{19}$$

$$\{x_N, y_{N-1}\} = \left\{1, \frac{2^{N-2}-1}{2^{N-2}}\right\} \tag{20}$$

are optimal; apart from situations (15), (18)–(20), there are no other pure strategy solutions in the duel by (17). If

$$a \in \left(0; \frac{1}{2^{N-2}-1}\right) \tag{21}$$

then optimal situation (15) is the single one. If

$$a \neq \frac{1}{2^{N-2}-1} \tag{22}$$

and (14) holds, optimal situation (13) is the single one. Finally, if neither (14) nor (16) holds, then the duel does not have a pure strategy solution.

The second particular case of the progressive LA1BSD was considered in [13], where

$$t_q = \sum_{n=1}^{q-1} \frac{1}{n(n+1)} = \frac{q-1}{q} \tag{23}$$

for $q = \overline{2, N-1}$.

This case was motivated by that the density of the duelist's pure strategies between $t_1 = 0$ and $t_N = 1$ grows too quickly if the geometrical progression by (7) is used. Progression (23) is smoother providing a sort of compactification of shooting moments. Meanwhile, article [13] proves that the solutions in the progressive LA1BSD (6) by (1)–(3), (23) for $N = 3$ are the same as the solutions in the progressive LA1BSD (6) by (1)–(3), (7) for $N = 3$. Besides, the progressive LA1BSD (6) by (1)–(3), (23) for $N \in \mathbb{N} \setminus \{1, 2, 3\}$ and a ≥ 1 has the single optimal situation (8), which coincides with the solution in the case of (7). Another coincidence is that in the case of (23) situation (8) is non-optimal by $a \in (0; 1)$ for $N \in \mathbb{N} \setminus \{1, 2\}$. The remaining results for (23) were proved [13] for $a \in (0; 1)$ and $N \in \mathbb{N} \setminus \{1, 2, 3\}$. Situation

$$\{x_3, y_3\} = \left\{\frac{2}{3}, \frac{2}{3}\right\} \tag{24}$$

is optimal only if $a = \frac{1}{2}$. Except for the third and last shooting moments $t_3 = \frac{2}{3}$ and $t_N = 1$, there are no other optimal pure strategies. The 4×4 duel with has four optimal pure strategy situations: situation (24) and situations

$$\{x_4, y_4\} = \{1, 1\}, \tag{25}$$

$$\{x_3, y_4\} = \left\{\frac{2}{3}, 1\right\}, \tag{26}$$

$$\{x_4, y_3\} = \left\{1, \frac{2}{3}\right\}. \tag{27}$$

Finally, situation (15) is single optimal for $N \in \mathbb{N} \setminus \{1, 2, 3, 4\}$ and

$$a \leqslant \frac{1}{N-2}. \tag{28}$$

In the 4×4 duel with

$$a < \frac{1}{N-2} = \frac{1}{2} \tag{29}$$

situation (15) is single optimal as well.

Despite progression (23) is smoother than progression (7), it still lacks a reasonable last-to-penultimate ratio

$$\frac{t_N}{t_{N-1}} = \frac{1}{t_{N-1}}, \tag{30}$$

which is

$$\frac{1}{t_{N-1}} = \frac{N-1}{N-2} \tag{31}$$

for (23), whereas ratio (30) is

$$\frac{1}{t_{N-1}} = \frac{2^{N-2}}{2^{N-2}-1} \tag{32}$$

for (7). Indeed,

$$\frac{N-1}{N-2} - \frac{2^{N-2}}{2^{N-2}-1} = \frac{N-2+1}{N-2} - \frac{2^{N-2}-1+1}{2^{N-2}-1} =$$

$$= 1 + \frac{1}{N-2} - 1 - \frac{1}{2^{N-2}-1} = \frac{1}{N-2} - \frac{1}{2^{N-2}-1} =$$
$$= \frac{2^{N-2} - 1 - N + 2}{(N-2)\left(2^{N-2}-1\right)} = \frac{2^{N-2} - N + 1}{(N-2)\left(2^{N-2}-1\right)} > 0 \quad (33)$$

for $N \in \mathbb{N} \setminus \{1, 2, 3\}$, where difference (33) between last-to-penultimate ratios (31) and (32) is 0 only at $N = 3$. So, in a duel with (23) the duelist gets a huge gap between the penultimate and last moment of possible shooting. Hence, another pattern of possible-shooting-moment progression is to be considered. According to this pattern, every next possible-shooting moment is obtained by adding the third of the remaining span to the current moment:

$$t_q = t_{q-1} + \frac{1 - t_{q-1}}{3} \quad (34)$$

for $q = \overline{2, N-1}$.

Herein, the local objective is to find pure strategy solutions of progressive LA1BSD (6) by (1)–(3), (34) for $N \in \mathbb{N} \setminus \{1, 2\}$.

3. Trivia and convention

Clearly, the most trivial duel size is 3×3. Its possible-shooting-moment progression is trivially a triple

$$T_3 = \{t_1, t_2, t_3\} = \left\{0, \frac{1}{3}, 1\right\}. \quad (35)$$

It is worth noting that the middle of the 3×3 duel time span is as twice as closer to the duel beginning than to the duel end.

Inasmuch as a pure strategy solution of duel (6) corresponds to a saddle point of skew-symmetric matrix (2) with entries (3), only a zero entry of this matrix can be a saddle point [7]. Therefore, a row containing a negative entry does not contain saddle points; neither does the respective column containing the positive entry. Hence, it is conventionally possible to conclude only on saddle points in definite rows of matrix (2), which imply the same conclusions on saddle points in respective columns.

It is rather trivial, but inasmuch as

$$k_{1j} = -ay_j < 0 \quad \forall\, j = \overline{2, N}$$

then the first row of matrix (2) with entries (3) is not an optimal strategy of the first duelist, and thus situation

$$\{x_1, y_1\} = \{0, 0\}$$

is never optimal in the duel. Another trivial remark is that a nonnegative row of matrix (2) with entries (3) contains a saddle point on the main diagonal of the matrix [7]. If a row contains only positive entries, except for the main diagonal entry, all the other $N - 1$ rows of the respective column contain negative entries, and thus this row contains a single saddle point which is the single one in the duel.

To study the duel in an easier way, pattern (34) of possible-shooting-moment progression ought to be represented similarly to (7) and (23), having the right-hand side term that depends only on q.

Theorem 1. Sequence (34) for (1) can be represented as

$$t_q = t_{q-1} + \frac{1 - t_{q-1}}{3} =$$
$$= \sum_{l=1}^{q-1} \frac{2^{l-1}}{3^l} = \frac{3^{q-1} - 2^{q-1}}{3^{q-1}} \quad (36)$$

for $q = \overline{2, N-1}$.

Proof. First, re-write (34) as

$$t_q = \frac{1 + 2t_{q-1}}{3} \quad (37)$$

for $q = \overline{2, N-1}$.

Equality (36), considered without its last term, can be proved by induction. In the base case, $q = 2$ and

$$t_2 = \sum_{l=1}^{1} \frac{2^{l-1}}{3^l} = \frac{1}{3}, \quad (38)$$

which is true by (35). By the inductive hypothesis it is assumed that equality (36), considered without its last term, holds for any $q = k$:

$$t_k = \sum_{l=1}^{k-1} \frac{2^{l-1}}{3^l}. \quad (39)$$

By the inductive step, it is about to show that equality (36), considered without its last term, holds for $q = k + 1$:

$$t_{k+1} = \sum_{l=1}^{k} \frac{2^{l-1}}{3^l}. \quad (40)$$

Moment t_{k+1} can be given by using (37):

$$t_{k+1} = \frac{1 + 2t_k}{3} = \frac{1}{3} + \frac{2}{3} \cdot \sum_{l=1}^{k-1} \frac{2^{l-1}}{3^l} =$$

$$=\frac{1}{3}+\sum_{l=1}^{k-1}\frac{2^l}{3^{l+1}}=\frac{2^0}{3^1}+\sum_{l=1}^{k-1}\frac{2^l}{3^{l+1}}=\sum_{j=1}^{k}\frac{2^{j-1}}{3^j}. \quad (41)$$

The last term in (41) coincides with the right-hand side term in (40). This proves equality

$$t_q=\sum_{l=1}^{q-1}\frac{2^{l-1}}{3^l} \quad (42)$$

for $q=\overline{2, N-1}$ by induction with (38) and (39), for the middle term in (36).

Equality

$$\sum_{l=1}^{q-1}\frac{2^{l-1}}{3^l}=\frac{3^{q-1}-2^{q-1}}{3^{q-1}} \quad (43)$$

for $q=\overline{2, N-1}$ is proved in the same way. In the base case, $q = 2$ and

$$\sum_{l=1}^{1}\frac{2^{l-1}}{3^l}=\frac{1}{3}=\frac{3-2}{3}=\frac{3^{2-1}-2^{2-1}}{3^{2-1}}, \quad (44)$$

which is true by (38). By the inductive hypothesis it is assumed that equality (43) holds for any $q = k$:

$$\sum_{l=1}^{k-1}\frac{2^{l-1}}{3^l}=\frac{3^{k-1}-2^{k-1}}{3^{k-1}}. \quad (45)$$

By the inductive step, it is about to show that equality (43) holds for $q = k + 1$:

$$\sum_{l=1}^{k}\frac{2^{l-1}}{3^l}=\frac{3^k-2^k}{3^k}. \quad (46)$$

The sum in the left-hand side of (46) can be represented as the sum of the right-hand side term in (45) and the k-th summand in the left-hand side of (46):

$$\sum_{l=1}^{k}\frac{2^{l-1}}{3^l}=\sum_{l=1}^{k-1}\frac{2^{l-1}}{3^l}+\frac{2^{k-1}}{3^k}=$$
$$=\frac{3^{k-1}-2^{k-1}}{3^{k-1}}+\frac{2^{k-1}}{3^k}=\frac{3^k-3\cdot 2^{k-1}}{3^k}+$$
$$+\frac{2^{k-1}}{3^k}=\frac{3^k-2\cdot 2^{k-1}}{3^k}=\frac{3^k-2^k}{3^k}. \quad (47)$$

The last term in (47) coincides with the right-hand side term in (46). This proves equality (43) by induction with (44) and (45). □

4. Three moments to shoot

Is the duel solution the same as for those two patterns of possible-shooting-moment progression, when the duelist has the fewest number of moments to shoot? The answer follows.

Theorem 2. Progressive LA1BSD (6) by (1)-(3), (36) for three moments to shoot ($N = 3$**)**

$$\langle X_3, Y_3, \mathbf{K}_3\rangle=\left\langle\left\{0, \frac{1}{3}, 1\right\}, \left\{0, \frac{1}{3}, 1\right\}, \mathbf{K}_3\right\rangle \quad (48)$$

has a single optimal situation (9) by $a\in(0;2)$**, a single optimal situation**

$$\{x_2, y_2\}=\left\{\frac{1}{3}, \frac{1}{3}\right\} \quad (49)$$

by $a > 2$**. At** $a = 2$ **this 4×4 duel has four optimal situations (49)**,

$$\{x_2, y_3\}=\left\{\frac{1}{3}, 1\right\}, \quad (50)$$

$$\{x_3, y_2\}=\left\{1, \frac{1}{3}\right\}, \quad (51)$$

and (9).

***Proof*.** Upon plugging elements of (35) into (3) for $N = 3$, the respective payoff matrix is

$$\mathbf{K}_3=\left[k_{ij}\right]_{3\times 3}=\begin{bmatrix} 0 & -\frac{a}{3} & -a \\ \frac{a}{3} & 0 & \frac{a}{3}(a-2) \\ a & -\frac{a}{3}(a-2) & 0 \end{bmatrix}. \quad (52)$$

If $a\in(0;2)$, matrix (52) has a single saddle point (9) due to the last row is positive except for main diagonal entry $k_{33} = 0$. If $a = 2$, the second and third rows are nonnegative, where

$$k_{22} = k_{23} = k_{32} = k_{33} = 0,$$

and matrix (52) has four saddle points: (49)–(51) and (9). If $a > 2$, matrix (52) has a single saddle point (49) due to the second row is positive except for main diagonal entry $k_{22} = 0$. □

Theorem 2 reads the difference between pattern (36) and patterns (7), (23), which lies in different second possible-shooting moments: it is $t_2=\frac{1}{2}$ being the middle of the duel time span for patterns (7), (23), whereas it is $t_2=\frac{1}{3}$ being the first third of the duel time span for pattern (36). Subsequently, duel solutions for pattern (36) differ from those for patterns (7), (23) in the boundary value of accuracy proportionality factor a, at which the solution changes. It is $a = 2$ for pattern (36), whereas it is $a = 1$ for patterns (7), (23). Structurally, however, all the three patterns have similar solutions for progressive 3×3 LA1BSDs: the last possible-shooting moment is the single optimal strategy for the accuracy proportionality factor below the boundary value; the second and last possible-shooting moments are only optimal strategies at the boundary value; the second possible-shooting moment is the single optimal strategy for the accuracy proportionality factor above the boundary value.

5. Second possible-shooting moment optimality

It is natural to conjecture that the boundary value of accuracy proportionality factor $a = 2$ must separate two cases of the duel solution just like value $a = 1$ separates those for patterns (7), (23). So, right below, 4×4 and bigger duels are considered by $a \geq 1$.

Theorem 3. Progressive LA1BSD (6) by (1)-(3), (36) for $N \in \mathbb{N} \setminus \{1, 2, 3\}$ and $a \geq 2$ has the single optimal situation (49)

Proof. Consider the second row of matrix (2), where

$$k_{21} = \frac{a}{3} > 0 \tag{53}$$

and

$$k_{2j} = \frac{a}{3} - ay_j + \frac{a^2}{3} y_j = \frac{a}{3} \cdot \left(1 - 3y_j + ay_j\right). \tag{54}$$

If $a = 2$ then

$$1 - 3y_j + ay_j = 1 - y_j \geqslant 0$$

and (54) is nonnegative:

$$k_{2j} = \frac{a}{3} \cdot \left(1 - 3y_j + ay_j\right) = \frac{2}{3} \cdot \left(1 - y_j\right) \geqslant 0 \quad \forall\ j = \overline{3, N}, \tag{55}$$

where $k_{2N} = 0$ is the second zero entry after k_{22} in the second row. Due to (53) and (55), situation (49) is a saddle point. However,

$$k_{N,N-1} = 2 \cdot 1 - 2 \cdot \frac{3^{N-2} - 2^{N-2}}{3^{N-2}} - 4 \cdot \frac{3^{N-2} - 2^{N-2}}{3^{N-2}} = 2 - 6 \cdot \frac{3^{N-2} - 2^{N-2}}{3^{N-2}} = 2 - 6 + 6 \cdot \frac{2^{N-2}}{3^{N-2}} = -4 + 6 \cdot \frac{2^{N-2}}{3^{N-2}} =$$

$$= 2 \cdot \left(3 \cdot \frac{2^{N-2}}{3^{N-2}} - 2\right) < 0 \tag{56}$$

due to

$$\frac{2^{N-3}}{3^{N-3}} < 1 \text{ and } 3 \cdot \frac{2^{N-2}}{3^{N-2}} < 2 \text{ for } N \in \mathbb{N} \setminus \{1, 2, 3\}.$$

Inequality (56) implies that the last row and last column of matrix (2) do not contain saddle points. So, situation (49) is single optimal by $a = 2$.

If $a > 2$ then it is sufficient to prove that

$$1 - 3y_j + ay_j > 0 \quad \forall\ j = \overline{3, N}. \tag{57}$$

Inequality (57), implying that the second row is positive except for main diagonal entry $k_{22} = 0$, is equivalent to inequality

$$a > \frac{3y_j - 1}{y_j} = 3 - \frac{1}{y_j} \quad \forall\ j = \overline{3, N} \tag{58}$$

by

$$\frac{1}{3} < y_j \leqslant 1 \quad \forall\ j = \overline{3, N}. \tag{59}$$

As (59) is true, then

$$3 > \frac{1}{y_j} \geqslant 1,$$

$$-3 < -\frac{1}{y_j} \leqslant -1,$$

$$0 < 3 - \frac{1}{y_j} \leqslant 2 < a, \tag{60}$$

whence inequality (60) directly implies that inequality (58) holds and situation (49) is single optimal by $a > 2$.

6. Second possible-shooting moment non-optimality

It was proved in [13, 15] that the second possible-shooting moment is not an optimal strategy by $0 < a < 1$ in progressive LA1BSDs (6) by (1)–(3) and $N \in \mathbb{N} \setminus \{1, 2\}$ for patterns (7) and (23). In those duels, noticeably, the second possible-shooting moment is the middle of the duel time span, unlike for pattern (36). See whether the similar property keeps for the LA1BSD with progressing-by-one-third shooting moments by (36), only by $0 < a < 2$ and the second possible-shooting moment being the first third of the duel time span.

Theorem 4. Situation (49) is never optimal in progressive LA1BSD (6) by (1)-(3), (36) for $N \in \mathbb{N} \setminus \{1, 2\}$ and $0 < a < 2$.

Proof. For $N \in \mathbb{N} \setminus \{1, 2\}$ consider the second row of matrix (2) whose last column entry

$$k_{2N} = \frac{a}{3} - a + \frac{a^2}{3} = \frac{a}{3} \cdot (a - 2) < 0 \qquad (61)$$

by $0 < a < 2$.

Inequality (61) directly implies that the second row of matrix (2) does not contain saddle points by $0 < a < 2$. □

7. Third possible-shooting moment optimality

In a 3×3 duel by $0 < a < 2$ it is optimal to shoot at the very last (third) possible-shooting moment (*Theorem 2*). The last possible-shooting moment is optimal for duelists in LA1BSDs for patterns (7) and (23) as well, but just by $a \in (0; 1)$. See whether the third possible-shooting moment in bigger LA1BSDs can be an optimal strategy for pattern (36).

Theorem 5. Progressive LA1BSD (6) by (1)-(3), (36) for $N \in \mathbb{N} \setminus \{1, 2, 3\}$ has an optimal pure strategy situation

$$\{x_3, y_3\} = \left\{\frac{5}{9}, \frac{5}{9}\right\} \qquad (62)$$

by

$$a \in \left[\frac{4}{5}; \frac{6}{5}\right] . \qquad (63)$$

Proof. Due to *Theorem 4*, situation (49) is not optimal, so the first two rows of matrix (2) do not contain saddle points. If situation

$$\{x_n, y_n\} = \left\{\frac{3^{n-1} - 2^{n-1}}{3^{n-1}}, \frac{3^{n-1} - 2^{n-1}}{3^{n-1}}\right\} \qquad (64)$$

by $n \in \left\{\overline{3, N-1}\right\}$ is optimal, then, in the n-th row of matrix (2), inequalities

$$k_{nj} = ax_n - ay_j - a^2 x_n y_j \geqslant 0 \quad \forall\, y_j < x_n \quad (\forall\, j = \overline{1, n-1}) \qquad (65)$$

and

$$k_{nj} = ax_n - ay_j + a^2 x_n y_j \geqslant 0 \quad \forall\, y_j > x_n \quad (\forall\, j = \overline{n+1, N}) \qquad (66)$$

must hold. From inequality (65) it follows that

$$\frac{x_n}{1 + ax_n} \geqslant y_j \quad \forall\, y_j < x_n \quad (\forall\, j = \overline{1, n-1}) \qquad (67)$$

As

$$y_j \leqslant \frac{3^{n-2} - 2^{n-2}}{3^{n-2}} < \frac{3^{n-1} - 2^{n-1}}{3^{n-1}} = x_n \quad \forall\, j = \overline{1, n-1} \qquad (68)$$

then inequality (67) is transformed into

$$\frac{3^{n-1} - 2^{n-1}}{3^{n-1}} \cdot \frac{1}{1 + a \cdot \frac{3^{n-1} - 2^{n-1}}{3^{n-1}}} \geqslant \frac{3^{n-2} - 2^{n-2}}{3^{n-2}},$$

$$\frac{3^{n-1} - 2^{n-1}}{3^{n-1} + a \cdot \left(3^{n-1} - 2^{n-1}\right)} \geqslant \frac{3^{n-2} - 2^{n-2}}{3^{n-2}},$$

$$3^{n-1} \cdot 3^{n-2} - 2^{n-1} \cdot 3^{n-2} \geqslant 3^{n-1} \cdot 3^{n-2} - 3^{n-1} \cdot 2^{n-2} + a \cdot \left(3^{n-1} - 2^{n-1}\right)\left(3^{n-2} - 2^{n-2}\right),$$

$$3^{n-1} \cdot 2^{n-2} - 2^{n-1} \cdot 3^{n-2} \geqslant a \cdot \left(3^{n-1} - 2^{n-1}\right)\left(3^{n-2} - 2^{n-2}\right),$$

$$3^{n-2} \cdot 2^{n-2} \cdot (3 - 2) \geqslant a \cdot \left(3^{n-1} - 2^{n-1}\right)\left(3^{n-2} - 2^{n-2}\right),$$

$$3^{n-2} \cdot 2^{n-2} \geqslant a \cdot \left(3^{n-1} - 2^{n-1}\right)\left(3^{n-2} - 2^{n-2}\right),$$

whence

$$a \leqslant \frac{3^{n-2} \cdot 2^{n-2}}{\left(3^{n-1} - 2^{n-1}\right)\left(3^{n-2} - 2^{n-2}\right)}. \qquad (69)$$

From inequality (66) it follows that

$$\frac{x_n}{1 - ax_n} \geqslant y_j \quad \forall\, y_j > x_n \quad (\forall\, j = \overline{n+1, N}) . \qquad (70)$$

As

$$1 \geqslant y_j > \frac{3^{n-1} - 2^{n-1}}{3^{n-1}} = x_n \quad \forall\, j = \overline{n+1, N}. \qquad (71)$$

then inequality (70) is transformed into

$$\frac{3^{n-1} - 2^{n-1}}{3^{n-1}} \cdot \frac{1}{1 - a \cdot \frac{3^{n-1} - 2^{n-1}}{3^{n-1}}} \geqslant 1,$$

$$\frac{3^{n-1} - 2^{n-1}}{3^{n-1} - a \cdot \left(3^{n-1} - 2^{n-1}\right)} \geqslant 1 . \qquad (72)$$

If

$$3^{n-1} - a \cdot \left(3^{n-1} - 2^{n-1}\right) > 0, \qquad (73)$$

i.e.

$$a < \frac{3^{n-1}}{3^{n-1} - 2^{n-1}} \ , \tag{74}$$

then inequality (72) is written as

$$3^{n-1} - 2^{n-1} \geqslant 3^{n-1} - a \cdot \left(3^{n-1} - 2^{n-1}\right),$$

whence

$$\frac{2^{n-1}}{3^{n-1} - 2^{n-1}} \leqslant a \, . \tag{75}$$

Therefore, situation (64) is optimal if inequality (69) holds along with inequalities (74) and (75). However,

$$\frac{3^{n-1}}{3^{n-1} - 2^{n-1}} - \frac{3^{n-2} \cdot 2^{n-2}}{\left(3^{n-1} - 2^{n-1}\right)\left(3^{n-2} - 2^{n-2}\right)} =$$

$$= \frac{3^{n-1} \cdot 3^{n-2} - 3^{n-1} \cdot 2^{n-2} - 3^{n-2} \cdot 2^{n-2}}{\left(3^{n-1} - 2^{n-1}\right)\left(3^{n-2} - 2^{n-2}\right)} =$$

$$= \frac{3^{n-1} \cdot 3^{n-2} - 4 \cdot 3^{n-2} \cdot 2^{n-2}}{\left(3^{n-1} - 2^{n-1}\right)\left(3^{n-2} - 2^{n-2}\right)} =$$

$$= \frac{3^{n-2} \cdot \left(3^{n-1} - 2^{n}\right)}{\left(3^{n-1} - 2^{n-1}\right)\left(3^{n-2} - 2^{n-2}\right)} > 0 \tag{76}$$

due to

$$3^{n-1} > 2^{n} \tag{77}$$

for $n \geqslant 3$.

Indeed, inequality (77) is true for $n = 3$:

$$3^2 = 9 > 8 = 2^3 \, .$$

Assume that inequality (77) holds for $n = k$:

$$3^{k-1} > 2^{k} \, . \tag{78}$$

For $n = k + 1$ inequality (77) turns into

$$3^{k} > 2^{k+1},$$

$$3 \cdot 3^{k-1} > 2 \cdot 2^{k},$$

$$\frac{3}{2} \cdot 3^{k-1} > 2^{k}, \tag{79}$$

whence inequality (79) holds due to inequality (78) holds. Inequality (76) means that

$$\frac{3^{n-2} \cdot 2^{n-2}}{\left(3^{n-1} - 2^{n-1}\right)\left(3^{n-2} - 2^{n-2}\right)} < \frac{3^{n-1}}{3^{n-1} - 2^{n-1}}$$

for $n \geq 3$ and thus it is sufficient to consider only stronger inequality (69), upon which weaker inequality (74) holds. Hence, situation (64) is optimal if inequality (69) holds along with inequality (75), i.e. if

$$a \in \left[\frac{2^{n-1}}{3^{n-1} - 2^{n-1}} ; \frac{3^{n-2} \cdot 2^{n-2}}{\left(3^{n-1} - 2^{n-1}\right)\left(3^{n-2} - 2^{n-2}\right)} \right]. \tag{80}$$

The difference between the right and left endpoints of the interval in membership (80) is:

$$\frac{3^{n-2} \cdot 2^{n-2}}{\left(3^{n-1} - 2^{n-1}\right)\left(3^{n-2} - 2^{n-2}\right)} - \frac{2^{n-1}}{3^{n-1} - 2^{n-1}} =$$

$$= \frac{3^{n-2} \cdot 2^{n-2} - 2^{n-1} \cdot 3^{n-2} + 2^{n-1} \cdot 2^{n-2}}{\left(3^{n-1} - 2^{n-1}\right)\left(3^{n-2} - 2^{n-2}\right)} =$$

$$= \frac{2^{n-1} \cdot 2^{n-2} + 3^{n-2} \cdot 2^{n-2} \cdot (1-2)}{\left(3^{n-1} - 2^{n-1}\right)\left(3^{n-2} - 2^{n-2}\right)} =$$

$$= \frac{2^{n-1} \cdot 2^{n-2} - 3^{n-2} \cdot 2^{n-2}}{\left(3^{n-1} - 2^{n-1}\right)\left(3^{n-2} - 2^{n-2}\right)} =$$

$$= \frac{2^{n-2} \cdot \left(2^{n-1} - 3^{n-2}\right)}{\left(3^{n-1} - 2^{n-1}\right)\left(3^{n-2} - 2^{n-2}\right)} \ . \tag{81}$$

Fraction (81) is nonnegative only for $n = 3$. Indeed, inequality

$$2^{n-1} > 3^{n-2} \qquad 82)$$

holds for $n = 3$ as

$$2^2 = 4 > 3 = 3^1,$$

but for $n = k$ there is inequality

$$2^{n-1} < 3^{n-2} \tag{83}$$

turning into

$$2^3 = 8 < 9 = 3^2,$$

and, assuming that for $n = k$ inequality (83) holds as

$$2^{k-1} < 3^{k-2}, \tag{84}$$

for $n = k + 1$ inequality (83) turns into

$$2^{k} < 3^{k-1},$$

$$2 \cdot 2^{k-1} < 3 \cdot 3^{k-2},$$

$$2^{k-1} < \frac{3}{2} \cdot 3^{k-2}, \tag{85}$$

whence inequality (85) holds due to inequality (84) holds. For $n = 3$ the interval in membership (80) turns into:

$$\left[\frac{2^2}{3^2-2^2};\frac{3^1\cdot 2^1}{\left(3^2-2^2\right)\left(3^1-2^1\right)}\right]=\left[\frac{4}{5};\frac{6}{5}\right].$$

Therefore, the duel has an optimal pure strategy situation (62) by (63). □

8. Last possible-shooting moment optimality

A corollary from *Theorem 5* is that 4×4 and bigger LA1BSDs by

$$a\in\left(0;\frac{4}{5}\right)\cup\left(\frac{6}{5};2\right) \tag{86}$$

do not have optimal pure strategy situations corresponding to all possible-shooting moments, except for the last one. The optimality of last-moment situation (15) is ascertained below for 5×5 and bigger LA1BSDs.

Theorem 6. In progressive LA1BSD (6) by (1)-(3), (36) for $N\in\mathbb{N}\setminus\{1,2,3,4\}$ and

$$a\in\left(0;\frac{2^{N-2}}{3^{N-2}-2^{N-2}}\right] \tag{87}$$

situation (15) is single optimal.

Proof. Situation (15) is optimal only if the last row of matrix (2) is nonnegative. Thus, the last, *N*-th, row of matrix (2) contains a saddle point if inequality

$$k_{Nj}=a-ay_j-a^2y_j\geqslant 0 \quad \forall\, y_j<1 \quad (\forall\, j=\overline{1,N-1}) \tag{88}$$

holds. It is easy to see in (88) that if inequality

$$1-y_{N-1}-ay_{N-1}\geqslant 0 \tag{89}$$

is true, then inequality (88) is true as well. From inequality (89) it follows that

$$\frac{1-y_{N-1}}{y_{N-1}}\geqslant a, \tag{90}$$

$$\frac{1-\dfrac{3^{N-2}-2^{N-2}}{3^{N-2}}}{\dfrac{3^{N-2}-2^{N-2}}{3^{N-2}}}\geqslant a, \tag{91}$$

$$\frac{2^{N-2}}{3^{N-2}-2^{N-2}}\geqslant a, \tag{92}$$

whence (87) implies optimality of situation (15). Meanwhile, inequality

$$\frac{2^{N-2}}{3^{N-2}-2^{N-2}}\leqslant\frac{8}{19}<\frac{4}{5} \tag{93}$$

holds for $N\in\mathbb{N}\setminus\{1,2,3,4\}$. Indeed, from inequality (93) it follows that

$$19\cdot 2^{N-2}\leqslant 8\cdot 3^{N-2}-8\cdot 2^{N-2},$$

$$27\cdot 2^{N-2}\leqslant 8\cdot 3^{N-2},$$

$$3^3\cdot 2^{N-2}\leqslant 2^3\cdot 3^{N-2},$$

whence

$$2^{N-5}\leqslant 3^{N-5}$$

for $N\in\mathbb{N}\setminus\{1,2,3,4\}$.

Inequality (93) implies that

$$\left(0;\frac{2^{N-2}}{3^{N-2}-2^{N-2}}\right]\subseteq\left(0;\frac{8}{19}\right]\subset\left(0;\frac{4}{5}\right) \tag{94}$$

for $N\in\mathbb{N}\setminus\{1,2,3,4\}$.

Membership (94) with the inclusion obeys membership (86), which implies that by (87) situation (15) in 5×5 and bigger LA1BSDs is single optimal. □

Inequality (93) is false for $N = 4$ as

$$\frac{2^2}{3^2-2^2}=\frac{4}{5}.$$

This leads to a specificity of 4×4 LA1BSDs by (87).

Theorem 7. In progressive 4×4 **LA1BSD (6) by (1)-(3), (36) for** $N = 4$ **and**

$$a\in\left(0;\frac{4}{5}\right) \tag{95}$$

situation (15) is single optimal. The 4×4 **LA1BSD by**

$$a=\frac{4}{5} \tag{96}$$

has four optimal pure strategy situations: (62),

$$\{x_3,y_4\}=\left\{\frac{5}{9},1\right\}, \tag{97}$$

$$\{x_4,y_3\}=\left\{1,\frac{5}{9}\right\}, \tag{98}$$

and (15).

Proof. Situation (15) is single optimal if the last, fourth, row of matrix (2) is positive, except for entry $k_{44}=0$. In *Theorem 6*, it follows from (88)–(92) for $N = 4$ that situation (15) is single optimal when

$$\frac{2^{N-2}}{3^{N-2}-2^{N-2}}=\frac{4}{5}>a, \tag{99}$$

i.e. if membership (95) is true. If (96) is true, situation (15) is optimal as well owing to (88)–(92) hold for $N = 4$. In addition, situation (62) is optimal in accordance with *Theorem 5* as membership (63) is also true. This additionally implies optimality of situations (97) and (98).□

9. Non-solvability in pure strategies

Just like the LA1BSDs for patterns (7) and (23), the LA1BSD for pattern (36) is not solved in pure strategies within a subset of values of the accuracy proportionality factor. This is proved by the two following assertions.

Theorem 8. Progressive 4×4 **LA1BSD (6) by (1)–(3), (36) for** $N = 4$ **is not solved in pure strategies by**

$$a \in \left(\frac{6}{5}; 2\right). \tag{100}$$

Proof. The 4×4 LA1BSD is solved in pure strategies by $a \geq 2$ (*Theorem 3*) and by (63) (*Theorem 5*) and by (95) (*Theorem 7*). By the remaining interval in (100), as the corollary from Theorem 5, the 4×4 LA1BSD does not have an optimal pure strategy situation that would contain three possible-shooting moments

$$\{t_1, t_2, t_3\} = \left\{0, \frac{1}{3}, \frac{5}{9}\right\}.$$

The last possible-shooting moment is non-optimal if, as a corollary from (88)–(93) in *Theorem 6* and *Theorem 7*, inequality (92) for $N = 4$ is false, i.e.

$$\frac{4}{5} < a, \tag{101}$$

which is true by (100). □

Theorem 9. Progressive LA1BSD (6) by (1)-(3), (36) for $N \in \mathbb{N} \setminus \{1, 2, 3, 4\}$ **is not solved in pure strategies by**

$$a \in \left(\frac{2^{N-2}}{3^{N-2} - 2^{N-2}}; \frac{4}{5}\right) \cup \left(\frac{6}{5}; 2\right). \tag{102}$$

Proof. Once again, the LA1BSD is solved in pure strategies by $a \geq 2$ (*Theorem 3*) and by (63) (*Theorem 5*) and by (87) (*Theorem 6*). Then, the corollary from *Theorem 5* and the corollary from *Theorem 6*, – particularly, with membership (94) and its inclusions, – is that the LA1BSD does not have optimal pure strategy situations by (102). □

It is easy to see that the subset in (102) of pure-strategy-solution non-existence expands as the duel becomes bigger.

Theorem 10. As the number of possible-shooting moments in progressive LA1BSD (6) by (1)-(3), (36) for $N \in \mathbb{N} \setminus \{1, 2, 3, 4\}$ **is increased, the last-moment-optimality interval by (87) shortens.**

Proof. This assertion means that

$$\lim_{N\to\infty} \left(0; \frac{2^{N-2}}{3^{N-2} - 2^{N-2}}\right] = \varnothing. \tag{103}$$

Consider a function

$$f(N) = \frac{2^{N-2}}{3^{N-2} - 2^{N-2}}. \tag{104}$$

The first derivative of function (104) is

$$\frac{df}{dN} = \frac{(\ln 2)\cdot 2^{N-2}\cdot\left(3^{N-2} - 2^{N-2}\right) - 2^{N-2}}{\left(3^{N-2} - 2^{N-2}\right)^2} \times$$

$$\times \frac{\left((\ln 3)\cdot 3^{N-2} - (\ln 2)\cdot 2^{N-2}\right)}{\left(3^{N-2} - 2^{N-2}\right)^2} =$$

$$= 2^{N-2} \cdot \frac{(\ln 2)\cdot 3^{N-2} - (\ln 2)\cdot 2^{N-2} - (\ln 3)}{\left(3^{N-2} - 2^{N-2}\right)^2} \times$$

$$\times \frac{3^{N-2} + (\ln 2)\cdot 2^{N-2}}{\left(3^{N-2} - 2^{N-2}\right)^2} =$$

$$= 2^{N-2}\cdot 3^{N-2}\cdot \frac{\ln 2 - \ln 3}{\left(3^{N-2} - 2^{N-2}\right)^2} < 0,$$

which means that (104) is a decreasing function of N. That is,

$$\lim_{N\to\infty} f(N) = \lim_{N\to\infty} \frac{2^{N-2}}{3^{N-2}\cdot\left(1 - \left(\frac{2}{3}\right)^{N-2}\right)} = 0,$$

whence (103) is true. □

Thus, as the duel becomes bigger, the non-constant interval in (102) becomes wider, expanding the accuracy subset of pure-strategy-solution non-existence towards

$$\left(0; \frac{4}{5}\right) \cup \left(\frac{6}{5}; 2\right) \tag{105}$$

for LA1BSDs with five and more possible-shooting moments.

10. Discussion and conclusion

Compared to the LA1BSDs for patterns (7) and (23), the LA1BSD with progressing-by-one-third shooting moments has a different boundary value of the accuracy proportionality factor, which separates two major cases of the duel solution. The LA1BSD for pattern (36) with four and more possible-shooting moments by $a \geq 2$ has the single optimal situation (49), according to which the duelist must shoot at the second moment being the first third of the duel time span (*Theorem 3*). When there are only three possible-shooting moments, the second moment is single optimal if $a > 2$, the last moment is single optimal if $a \in (0; 2)$, the second and last moments are both optimal if $a = 2$ (*Theorem 2*).

When $a \in (0; 2)$ and there are three or more possible-shooting moments, the second moment is never optimal for the duelist (*Theorem 4*). This is the case, where LA1BSD with progressing-by-one-third shooting moments significantly differs (in terms of its solution) from the LA1BSDs for patterns (7) and (23). Thus, in LA1BSDs for pattern (36) with four and more possible-shooting moments third moment $t_3 = \frac{5}{9}$ is optimal by (63) (*Theorem 5*), whereas third moment $t_3 = \frac{2}{3}$ is optimal in the LA1BSD for compactified-moments pattern (23) only if $a = \frac{1}{2}$ [13]. In the LA1BSD for geometrical-progression pattern (7) third moment $t_3 = \frac{3}{4}$ is particularly optimal if

$$a \in \left[\frac{1}{3}; \frac{2}{3}\right],$$

although optimality of later possible-shooting moments is also possible [15].

Just like for patterns (7) and (23), the last moment can be optimal in LA1BSDs for pattern (36) with four and more possible-shooting moments by sufficiently low values of the accuracy proportionality factor. The last moment is optimal if (87) is true (*Theorems 6* and *7*), where the length of the interval in (87) exponentially-like shortens as the number of possible-shooting moments (the size of the duel) is increased (*Theorem 10*). This last-moment-optimality interval shortening exists for patterns (7) and (23) as well, whose right endpoints in the interval are

$$\frac{1}{2^{N-2}-1}$$

and

$$\frac{1}{N-2},$$

respectively. Last-moment-optimality solutions of LA1BSDs for patterns (7), (23), and (36) with exactly four possible-shooting moments cannot be seamlessly surveyed. The 4×4 LA1BSD for geometrical-progression pattern (7) is not specifically distinguished from bigger LA1BSDs. Unlike LA1BSDs with the faster converging possible-shooting moments by (7), the duelist in the 4×4 LA1BSD with compactified shooting moments by (23) and $a \in \left(0; \frac{1}{2}\right)$ has the single optimal strategy to shoot at the duel very end. If the accuracy proportionality factor is equal to $\frac{1}{2}$, then the duelist in the 4×4 LA1BSD for pattern (23) possesses two optimal pure strategies $t_3 = \frac{2}{3}$ and $t_4 = 1$. This resembles the optimal behavior of the duelist in the 4×4 LA1BSD for pattern (36) and (96), where it is optimal to shoot at either $t_3 = \frac{5}{9}$ or $t_4 = 1$. If (95) is true, the last moment remains single optimal (*Theorem 7*).

Unlike the LA1BSD for pattern (23), which is not solved in pure strategies if

$$a \in \left(\frac{1}{N-2}; 1\right) \backslash \left\{\frac{1}{2}\right\} \qquad (106)$$

for $N \in \mathbb{N} \backslash \{1, 2, 3\}$ and the interval in (106) approaches to open interval (0;1) as the number of possible-shooting moments is increased, the LA1BSD for pattern (36) does not have a pure strategy solution by (102) (*Theorem 9*), i.e. there is a stable infinite subset of values of the accuracy proportionality factor below the boundary value $a = 2$ such that a pure strategy solution exists – see (63) and *Theorem 5*. This subset, whose length is $\frac{2}{5}$ comprising 20 % of the below-boundary-value interval, changes into interval

$$\left(0; \frac{6}{5}\right] \qquad (107)$$

in a 4×4 LA1BSD for pattern (36) (*Theorems 7* and *8*). Interval (107) comprises 60 % of the below-boundary-value interval (0;2).

The proved assertions contribute another specificity of the progressing-by-one-third shooting moments in LA1BSDs to the games of timing. Compared to LA1BSDs for patterns (7) and (23), the specificity consists in that the LA1BSD for pattern (36) has a constant interval of lower (weaker) shooting accuracies, at which the duelist possesses an optimal pure strategy. This interval is $\left[\frac{4}{5}; \frac{6}{5}\right]$ that symmetrically breaks the low-accuracy interval (0;2).

LA1BSDs with progressing-by-one-third shooting moments can be further studied for some nonlinearities in the accuracy function. For instance, it can be the quadratic accuracy as a case of the low-accurate duelist [10]. For a case of the high-accurate duelist, it is the square-root accuracy. Besides, the case of a value of the jitter added to progressing-by-one-third shooting moments, apart from the duel beginning and end time moments, can be considered [18].

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БЕЗШУМНА ДУЕЛЬ З ОДНІЄЮ КУЛЕЮ ЛІНІЙНОЇ ВЛУЧНОСТІ ТА ПРОГРЕСУЮЧИМИ НА ОДНУ ТРЕТИНУ МОМЕНТАМИ ПОСТРІЛУ

Проблематика. Розглянуто скінченну гру з нульовою сумою, яка моделює конкуруючу взаємодію між двома суб'єктами. Суб'єкт, якого ще називають дуелянтом, має виконати якусь дію (або, висловлюючись метафорично, здійснити постріл однією кулею) протягом стандартизованого проміжку часу, де куля може бути випущена лише у зазначені моменти часу. Для більш реалістичного симулювання взаємодії між дуелянтами кількість таких моментів можливого пострілу приймають скінченною, внаслідок чого гра (або ж дуель) стає дискретною. Для дуелянта залишається невідомим до кінця дуелі, чи інший дуелянт здійснив постріл і коли він відбувся. Дуелянт може не стріляти аж до самого кінця дуелі, але тоді постріл одначе здійснюється автоматично у цей кінцевий момент часу, оскільки дія має бути виконана у будь-якому випадку. Дуелянт виграє від здійснення пострілу якомога пізніше, але лише тоді, коли він випередить іншого дуелянта.

Мета дослідження. Мета полягає у тому, щоб для деякої моделі дискретної прогресії дуелі визначити оптимальну поведінку дуелянтів, за якої напруга збільшується з наближенням кінця дуелі та з'являється більше можливостей для пострілу.

Методика реалізації. Обидва дуелянти діють за тих самих умов, тому ця безшумна дуель з однією кулею є симетричною. Відтак оптимальне значення гри дорівнює 0, і дуелянти мають однакові оптимальні стратегії. Влучність пострілу є лінійною і визначається коефіцієнтом пропорційності точності.

Результати дослідження. Усі розв'язки у чистих стратегіях для таких дуелей знайдені залежно від цього коефіцієнта, де моменти можливого пострілу складають модель деякої прогресії. Згідно з цією моделлю кожний наступний момент можливого пострілу отримують додаванням третини часового проміжку, що залишається до кінця дуелі. Розв'язки для цієї моделі порівнюються з відомими розв'язками для моделі геометричної прогресії, а також моделі, в якій моменти можливого пострілу прогресують більш помірно.

Висновки. Доведені твердження розкривають ще одну особливість прогресуючих на одну третину моментів пострілу у безшумних дуелях з однією кулею лінійної влучності у класі часових ігор. Якщо порівнювати дуелі з іншими моделями дискретної прогресії, ця особливість полягає у тому, що дуель із прогресуючими на одну третину моментами пострілу має постійний інтервал нижніх (слабших) влучностей, за яких дуеліст має оптимальну чисту стратегію. Цим інтервалом є $\left[\frac{4}{5}; \frac{6}{5}\right]$, який симетрично розбиває інтервал (0; 2) слабкої влучності.

Ключові слова: безшумна дуель з однією кулею; лінійна влучність; матрична гра; розв'язок у чистих стратегіях; прогресуючі на одну третину моменти пострілу.

Рекомендована Радою
факультету прикладної математики
КПІ ім. Ігоря Сікорського

Надійшла до редакції
18 серпня 2025 року

Прийнята до публікації
08 грудня 2025 року