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Vadim Romanuke*
Vinnytsia Institute of Trade and Economics of State University of Trade and Economics, Vinnytsia, Ukraine
*Corresponding author: romanukevadimv@gmail.com

# TIME-UNIT SHIFTING IN 3-PERSON GAMES IN FINITE AND UNCOUNTABLY INFINITE STAIRCASE-FUNCTION SPACES SOLVED IN PURE STRATEGIES

**Background.** Games played with staircase-function pure strategies can model discrete-time dynamics of rationalizing the distribution of some limited resources among players. Along with 2-person games, 3-person games are the most applicable models of rationalization in economics, ecology, social sciences, politics, government, and sports. There is a known method of finding an equilibrium in a 3-person game played in staircase-function pure strategy spaces. The time interval on which the game is defined consists of an integer number of time units. The equilibrium is stacked from time-unit equilibria. An open problem is a multiplicity of equilibria (on some time units) leading to a multiplicity of equilibrium stacks. Another open question is how to deal with a 3-person game in which the time interval can be changed or shifted by an integer number of time units.

**Objective.** The purpose of the paper is to expand and develop the tractable method of solving 3-person games played within players' finite sets of staircase functions for the case when the length of the time interval on which the 3-person game is defined is varied by an integer number of time units.

**Methods.** To achieve the said objective, a 3-person game, in which the players' strategies are staircase functions of time, is formalized. In such a game, the set of the player's pure strategies is a continuum of staircase functions. The time can be thought of as it is discrete due to the time interval is comprised of time units (subintervals). Then the set of possible values of the player's pure strategy is discretized so that the player possesses a finite set of staircase functions.

**Results.** The known method is expanded to build a single pure-strategy equilibrium stack in a discrete-time staircase-function 3-person game. The criterion for selecting a single equilibrium solution is to maximize the players' payoffs sum. In the case of a time-unit shifting, this criterion allows extracting the respective best staircase-function equilibrium pure strategy of the player in any "narrower" subgame from the player's best staircase-function equilibrium pure strategy in the "wider" game.

**Conclusions.** A tractable and efficient method of finding the best pure-strategy equilibrium in a 3-person game played in finite or uncountably infinite staircase-function spaces is to solve a succession of time-unit 3-person games, whereupon their best equilibria are stacked into the best pure-strategy equilibrium. To deal with the case when not every time-unit 3-person game is solved in pure strategies, an effective way is to put a staircase-function game on hold-up on those time units which do not have pure-strategy equilibria. The result of putting the staircase-function game on hold-ups is that the player will obtain one's best staircase-function equilibrium pure strategy with gaps, whichever the time interval and time-unit shifting are.

**Keywords:** game theory; payoff functional; 3-person game; staircase-function strategy; trimatrix game; staircase-function equilibrium pure strategy.

## Introduction

In practical tasks, noncooperative 3-person games are well-posed and easy-to-interpret models to rationalize the distribution of real-world resources, funds, energy, facilities, tools, etc. [1, 2]. Along with 2-person games, 3-person games are the most applicable models of rationalization in economics [2, 3], ecology [2, 4, 5], social sciences [6], politics [7], government [3, 8], sports [8, 9]. Even games in which players possess just two pure strategies have a good practical impact. For instance, a problem of rationalizing industrial wastewater treatment considered in [4, 5, 10, 11] is solved by using dyadic 3-person games. As industrial enterprises may violate conventions about water treatment, they are

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fined. The fines are directed to control water pollution by measuring it and treating wastewater additionally, if necessary. However, an enterprise may reduce or stop its manufacturing under threat of heavy fines. This results in a budget cut for water resources conservation and recirculation. The dyadic 3-person game models a process of balancing the fines. As a result, the balancing allows industrial enterprises to keep functioning along with satisfactory water recovery.

In particular, the dyadic 3-person game solution in [4] was searched in the form of equilibrium on a regular finite lattice of situations obtained by sampling the continuous set of those situations. An approximate solution was found using concessions in the equilibrium, where the cost for water treatment system application was a conventional unit for each enterprise per a period of time (a day, a week, or a month). By that solution, the water treatment system is turned off for 3 periods of 10, and the 2-fine (a range of the fine for when only two enterprises do simultaneously not treat wastewater) is optimally set at 0.34 units, whereas the 3-fine (a range of the fine for when no one treats wastewater) is set at 1.394 units. Obviously, switching from "clean" to "polluting" manufacturing and backwards can be controlled once per those 10 periods under the corresponding water treatment schedule. In such a schedule, the enterprise develops one's metastrategy which, in fact, appears to be a primitive staircase function of time.

Games played with staircase-function pure strategies have been recently studied in [12, 13] and, in a more peculiar way, [14, 15]. Whereas an ordinary ("classical") pure strategy of the player is a simple (point) action whose duration is usually negligible and represented as just a (time) point, a staircase-function pure strategy is a complex process comprising a series of simple actions (moves, decisions, changes, strikes, etc.). The staircase-function pure strategy is defined on a time interval. The time interval is broken into a set of time subintervals (units), on which the strategy is (approximately considered) constant. In fact, a pure staircase-function strategy can be considered as an ordinary mixed strategy unfolded over the time interval. A mixed staircase-function strategy is a far more complicated case, where at least a unit corresponds to an ordinary mixed strategy. The composition of ordinary pure and ordinary mixed strategies that a player has to switch through time units is a model whose practical implementation requires a definitely great number of game repetitions [4, 5, 10, 11, 16, 17].

If each of the three players possesses a finite number of staircase-function pure strategies, the respective 3-person game is finite. The finite 3-person game can be represented as a trimatrix game, whichever pure strategy form is. If pure strategies are staircase functions (of time whose duration is broken into time subintervals or time units), the respective finite 3-person game can be called the trimatrix staircase-function game [1, 13]. Clearly, the number of pure-strategy situations in a trimatrix staircase-function game grows immensely as the number of time units ("stair" subintervals) increases, or the number of possible values of the player's pure strategy increases, or they both increase [14, 18, 19]. For instance, if the number of time units is just 4, and the number of possible values of every player's pure strategy is 8, then there is a finite set of

$$8^4 = 4096$$

possible pure strategies (i.e., 4-subinterval staircase functions of time) at this player. The respective trimatrix staircase-function game has a size of

$$4096 \times 4096 \times 4096,$$

and so there are

$$8^4 \cdot 8^4 \cdot 8^4 = 4096 \cdot 4096 \cdot 4096 = 68\,719\,476\,736$$

pure-strategy situations (more than 68.7 billion ones) in this game. If an additional time unit is (somehow) included, the game size increases dramatically: there are

$$8^5 \cdot 8^5 \cdot 8^5 = 32768 \cdot 32768 \cdot 32768 = 35\,184\,372\,088\,832$$

pure-strategy situations (more than 35.1 trillion ones!) in the respective

trimatrix $32768 \times 32768 \times 32768$ game.

Obviously, solving 3-person games of such gigantic sizes is intractable, let alone that there is no universal algorithm for solving any finite 3-person game played with ordinary ("classical") pure and mixed strategies [19, 20]. Solving 3-person games in staircase-function pure strategies is always possible, but it takes too much computational resources even if there are a few time units.

For the equilibrium solution type, a method of solving a 3-person game played in staircase-function pure strategy spaces was presented in [13]. The spaces can be finite and uncountably infinite (continuous). The method is based on stacking equilibria of "short" 3-person games, each defined on a time unit where the pure strategy value is constant. In the case of finite 3-person games, the stack is any interval-wise combination (succession) of the respective equilibria of the "short" trimatrix games (including equilibria in mixed strategies). Unlike the straightforward approach to solving directly the "long" 3-person game (finite or infinite), the presented method "breaks" the "long" game into a succession of "short" games, making thus its solving tractable. However, an open problem is a multiplicity of equilibria (on some time units) leading to a multiplicity of equilibrium stacks. Another open question is how to deal with a 3-person game in which the function-strategy can be redefined on a changed time interval (e.g., shifted by an integer number of time units) [21, 22]. For instance, if the pure staircase-function strategy is indeed considered as an ordinary mixed strategy unfolded over a time interval, but the number of game rounds is shortened, the time interval should become shorter [13, 16, 18, 23, 24]. Such game-interval modifications may occur frequently, and respective game solutions must be found even faster.

**Problem statement**

Reasoning from the mentioned issues, the objective is to expand and develop the tractable method of solving 3-person games played within players' finite sets of staircase functions [13, 14] for the case when the length of the time interval on which the 3-person game is defined is varied by an integer number of time units. The solution type is equilibrium in staircase-function pure strategies, where it is presumed that such an equilibrium exists. The case when the player possesses an uncountably infinite set (space) of staircase functions is to be considered as well. To meet the objective, the following eight tasks are to be fulfilled:

1. To formalize a 3-person game, in which the players' strategies are functions (of time) defined on a time interval. In such a game, the set of the player's pure strategies is a continuum of functions. Such function-strategies are presumed to be bounded and Lebesgue-integrable.
2. To formalize a 3-person game, in which the players' strategies are staircase functions defined on the time interval. In such a game, the set of the player's pure strategies is a continuum of staircase functions (of time). The time can be thought of as it is discrete due to the time interval is comprised of time units (subintervals).
3. To discretize the set of possible values of the player's pure strategy so that the game played with staircase-function strategies be defined on a product of staircase-function finite spaces.
4. To expand and develop the known method of solving 3-person games (the solution of the pure-strategy equilibrium type) played in staircase-function finite and uncountably infinite spaces by considering a possibility of narrowing the time interval on which the 3-person game is defined. In addition, a method of selecting a single pure-strategy equilibrium should be suggested.
5. To suggest a way of solution when not every "short" 3-person game is solved in pure strategies.
6. To give an example of how the suggested method is applied.
7. To discuss practical applicability and scientific significance of the method for the game theory and operations research.
8. To make an unbiased conclusion on the contribution to the game theory field. An outlook of how the study might be further developed is to be made as well.

**3-person game played with pure strategies as functions of time**

In a 3-person game, in which the player's pure strategy is a function of time, each of the players uses time-varying strategies defined almost everywhere on interval $[t_1; t_2]$ by $t_2 > t_1$. Pure strategies of the first, second, and third players denoted by $x(t)$, $y(t)$, and $z(t)$, respectively, are presumed to be bounded. Besides, the square of the function-strategy is presumed to be Lebesgue-integrable [25]. Thus, pure strategies of the player belong to a rectangular functional space of functions of time:

$$X = \{x(t),\ t \in [t_1; t_2],\ t_1 < t_2 : a_{\min} \leqslant x(t) \leqslant a_{\max} \text{ by } a_{\min} < a_{\max}\} \subset \mathbb{L}_2[t_1; t_2] \quad (1)$$

and

$$Y = \{y(t),\ t \in [t_1; t_2],\ t_1 < t_2 : b_{\min} \leqslant y(t) \leqslant b_{\max} \text{ by } b_{\min} < b_{\max}\} \subset \mathbb{L}_2[t_1; t_2] \quad (2)$$

and

$$Z = \{z(t),\ t \in [t_1; t_2],\ t_1 < t_2 : c_{\min} \leqslant z(t) \leqslant c_{\max} \text{ by } c_{\min} < c_{\max}\} \subset \mathbb{L}_2[t_1; t_2] \quad (3)$$

are the sets (sometimes referred to as action spaces) of the players' pure strategies.

The player's payoff in situation

$$\{x(t),\ y(t),\ z(t)\} \quad (4)$$

is presumed to be an integral functional [12, 13, 14, 15, 25]. Thus, the first, second, and third players' payoffs in situation (4) are

$$F\left(x(t),\, y(t),\, z(t)\right) = \int\limits_{[t_1;\, t_2]} f\left(x(t),\, y(t),\, z(t),\, t\right) d\mu(t), \tag{5}$$

$$G\left(x(t),\, y(t),\, z(t)\right) = \int\limits_{[t_1;\, t_2]} g\left(x(t),\, y(t),\, z(t),\, t\right) d\mu(t), \tag{6}$$

$$H\left(x(t),\, y(t),\, z(t)\right) = \int\limits_{[t_1;\, t_2]} h\left(x(t),\, y(t),\, z(t),\, t\right) d\mu(t), \tag{7}$$

respectively, where

$$f\left(x(t),\, y(t),\, z(t),\, t\right), \tag{8}$$

$$g\left(x(t),\, y(t),\, z(t),\, t\right), \tag{9}$$

$$h\left(x(t),\, y(t),\, z(t),\, t\right) \tag{10}$$

are functions of $x(t)$, $y(t)$, $z(t)$, explicitly including time $t$. Therefore, a 3-person game

$$\left\langle \{X, Y, Z\}, \left\{F\left(x(t),\, y(t),\, z(t)\right),\right.\right. \\ \left.\left. G\left(x(t),\, y(t),\, z(t)\right),\, H\left(x(t),\, y(t),\, z(t)\right)\right\}\right\rangle \tag{11}$$

is uncountably infinite due to it is defined on product

$$X \times Y \times Z \subset \mathbb{L}_2\left[t_1;\, t_2\right] \times \mathbb{L}_2\left[t_1;\, t_2\right] \times \mathbb{L}_2\left[t_1;\, t_2\right] \tag{12}$$

of uncountably infinite rectangular functional spaces (1)–(3) of players' pure strategies. An example of a situation in 3-person game (11) is given in Fig. 1. There are no restrictions to the strategy form the players can use. In the example, the first player uses a sinusoidal strategy with an exponential growth, the second player uses a curvilinear strategy, and the third player uses a close-to-straight ascending line strategy. Each of sets (1)–(3) is a continuum of functions including a subset of staircase functions (this subset is a continuum as well).

In general, the player's payoff functional may have a terminal component. Thus, instead of (5)–(7), the players' payoffs in situation (4) may be calculated as

$$F\left(x(t),\, y(t),\, z(t)\right) = \int\limits_{[t_1;\, t_2]} f\left(x(t),\, y(t),\, z(t),\, t\right) d\mu(t) + T_f\left(x(t_2),\, y(t_2),\, z(t_2),\, t_2\right), \tag{13}$$

$$G\left(x(t),\, y(t),\, z(t)\right) = \int\limits_{[t_1;\, t_2]} g\left(x(t),\, y(t),\, z(t),\, t\right) d\mu(t) + T_g\left(x(t_2),\, y(t_2),\, z(t_2),\, t_2\right), \tag{14}$$

$$H\left(x(t),\, y(t),\, z(t)\right) = \int\limits_{[t_1;\, t_2]} h\left(x(t),\, y(t),\, z(t),\, t\right) d\mu(t) + T_h\left(x(t_2),\, y(t_2),\, z(t_2),\, t_2\right) \tag{15}$$

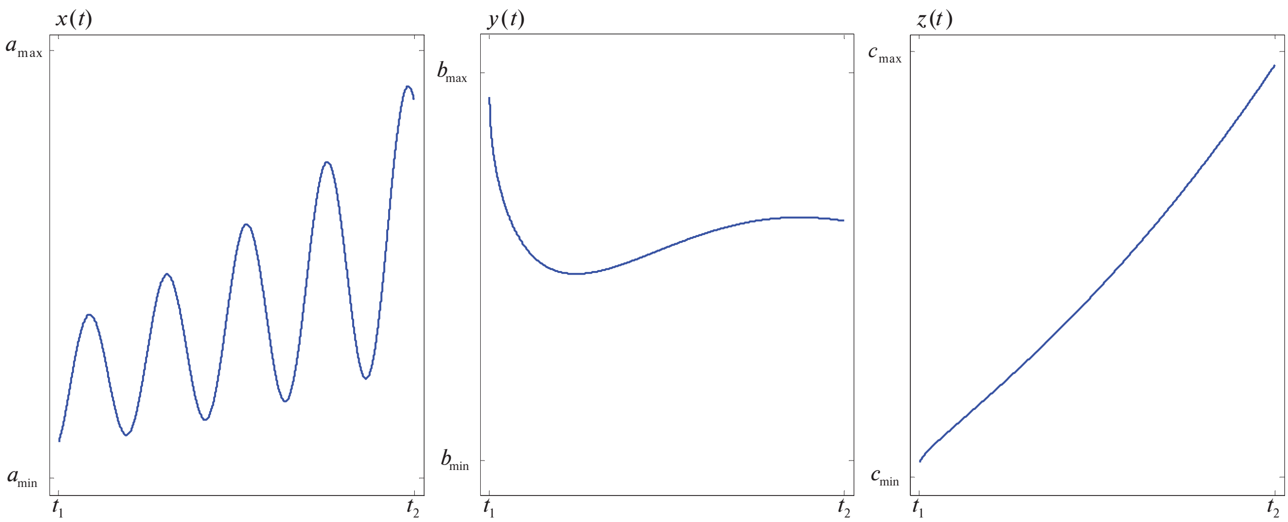


Fig. 1. A situation (4) in 3-person game (11) played in uncountably infinite functional spaces (1)–(3)

by some terminal functions [26]

$$T_f\left(x(t_2),\, y(t_2),\, z(t_2),\, t_2\right), \tag{16}$$

$$T_g\left(x(t_2),\, y(t_2),\, z(t_2),\, t_2\right), \tag{17}$$

$$T_h\left(x(t_2),\, y(t_2),\, z(t_2),\, t_2\right), \tag{18}$$

depending on only the final state of the player's strategy, but this case is not to be considered here.

### 3-person game played with staircase-function strategies through discrete time

Presume that the players' pure strategies in game (11) can change their values only a finite number of times. Denote by $N$ the number of time units (subintervals) at which the player's pure strategy is constant, where $N \in \mathbb{N} \setminus \{1\}$. Then the player's pure strategy is a staircase function having at most $N$ different values. Let there be a time-interval breaking

$$\Theta = \left\{t_1 = \tau^{(0)} < \tau^{(1)} < \tau^{(2)} < \ldots < \tau^{(N-1)} < \tau^{(N)} = t_2\right\}, \tag{19}$$

where $\{\tau^{(i)}\}_{i=1}^{N-1}$ are time points at which the staircase-function strategy can change its value. Generally speaking, time-interval breaking (19) is not equidistant, although in most practical cases it is equidistant, i.e.

$$\tau^{(i)} - \tau^{(i-1)} = \frac{t_2 - t_1}{N} \quad \forall\, i = \overline{1,\, N}. \tag{20}$$

The staircase-function strategies are right-continuous [12, 13, 14, 15, 25]:

$$\lim_{\substack{\varepsilon>0\\ \varepsilon\to 0}} x\left(\tau^{(i)} + \varepsilon\right) = x\left(\tau^{(i)}\right), \tag{21}$$

$$\lim_{\substack{\varepsilon>0\\ \varepsilon\to 0}} y\left(\tau^{(i)} + \varepsilon\right) = y\left(\tau^{(i)}\right), \tag{22}$$

$$\lim_{\substack{\varepsilon>0\\ \varepsilon\to 0}} z\left(\tau^{(i)} + \varepsilon\right) = z\left(\tau^{(i)}\right) \tag{23}$$

for $i = \overline{1,\, N-1}$, whereas (if the strategy value changes)

$$\lim_{\substack{\varepsilon>0\\ \varepsilon\to 0}} x\left(\tau^{(i)} - \varepsilon\right) \neq x\left(\tau^{(i)}\right), \tag{24}$$

$$\lim_{\substack{\varepsilon>0\\ \varepsilon\to 0}} y\left(\tau^{(i)} - \varepsilon\right) \neq y\left(\tau^{(i)}\right), \tag{25}$$

$$\lim_{\substack{\varepsilon>0\\ \varepsilon\to 0}} z\left(\tau^{(i)} - \varepsilon\right) \neq z\left(\tau^{(i)}\right) \tag{26}$$

for $i = \overline{1,\, N-1}$. In the end time point, obviously,

$$\lim_{\substack{\varepsilon>0\\ \varepsilon\to 0}} x\left(\tau^{(N)} - \varepsilon\right) = x\left(\tau^{(N)}\right), \tag{27}$$

$$\lim_{\substack{\varepsilon>0\\ \varepsilon\to 0}} y\left(\tau^{(N)} - \varepsilon\right) = y\left(\tau^{(N)}\right), \tag{28}$$

$$\lim_{\substack{\varepsilon>0\\ \varepsilon\to 0}} z\left(\tau^{(N)} - \varepsilon\right) = z\left(\tau^{(N)}\right). \tag{29}$$

A 3-person game played with staircase-function strategies through discrete time can be defined by using (1)—(12), (19)—(29).

**Definition 1.** 3-person game (11) defined on product (12) of rectangular functional spaces (1)—(3) is called a discrete-time staircase-function 3-person game by time-interval breaking (19), if (21)—(29) hold and

$$x(t) = \alpha_i \in \left[a_{\min};\, a_{\max}\right],$$
$$y(t) = \beta_i \in \left[b_{\min};\, b_{\max}\right],$$
$$z(t) = \gamma_i \in \left[c_{\min};\, c_{\max}\right] \quad \forall\, t \in \left[\tau^{(i-1)};\, \tau^{(i)}\right)$$
$$\text{for } i = \overline{1,\, N-1} \text{ and } x(t) = \alpha_N \in \left[a_{\min};\, a_{\max}\right],$$
$$y(t) = \beta_N \in \left[b_{\min};\, b_{\max}\right],$$
$$z(t) = \gamma_N \in \left[c_{\min};\, c_{\max}\right] \quad \forall\, t \in \left[\tau^{(N-1)};\, \tau^{(N)}\right], \tag{30}$$

where the factual payoff of the first player in situation

$$\left\{\alpha_i,\, \beta_i,\, \gamma_i\right\} \tag{31}$$

is

$$F_i(\alpha_i,\, \beta_i,\, \gamma_i) = \int\limits_{\left[\tau^{(i-1)};\, \tau^{(i)}\right)} f(\alpha_i,\, \beta_i,\, \gamma_i,\, t)\, d\mu(t) \quad \forall\, i = \overline{1,\, N-1} \tag{32}$$

and

$$F_N(\alpha_N,\, \beta_N,\, \gamma_N) = \int\limits_{\left[\tau^{(N-1)};\, \tau^{(N)}\right]} f(\alpha_N,\, \beta_N,\, \gamma_N,\, t)\, d\mu(t), \tag{33}$$

the factual payoff of the second player in situation (31) is

$$G_i(\alpha_i, \beta_i, \gamma_i) = \int_{[\tau^{(i-1)}; \tau^{(i)})} g(\alpha_i, \beta_i, \gamma_i, t) d\mu(t) \quad \forall\, i = \overline{1, N-1} \tag{34}$$

and

$$G_N(\alpha_N, \beta_N, \gamma_N) = \int_{[\tau^{(N-1)}; \tau^{(N)}]} g(\alpha_N, \beta_N, \gamma_N, t) d\mu(t), \tag{35}$$

the factual payoff of the third player in situation (31) is

$$H_i(\alpha_i, \beta_i, \gamma_i) = \int_{[\tau^{(i-1)}; \tau^{(i)})} h(\alpha_i, \beta_i, \gamma_i, t) d\mu(t) \quad \forall\, i = \overline{1, N-1} \tag{36}$$

and

$$H_N(\alpha_N, \beta_N, \gamma_N) = \int_{[\tau^{(N-1)}; \tau^{(N)}]} h(\alpha_N, \beta_N, \gamma_N, t) d\mu(t). \tag{37}$$

Situation (4) in the discrete-time staircase-function 3-person game is a stack of successive situations

$$\{\{\alpha_i, \beta_i, \gamma_i\}\}_{i=1}^{N} \tag{38}$$

in a succession of $N$ (ordinary) 3-person games

$$\langle \{[a_{\min}; a_{\max}], [b_{\min}; b_{\max}], [c_{\min}; c_{\max}]\}, \{F_i(\alpha_i, \beta_i, \gamma_i), G_i(\alpha_i, \beta_i, \gamma_i), H_i(\alpha_i, \beta_i, \gamma_i)\} \rangle \quad \text{for} \quad i = \overline{1, N} \tag{39}$$

defined on product

$$[a_{\min}; a_{\max}] \times [b_{\min}; b_{\max}] \times [c_{\min}; c_{\max}] \tag{40}$$

by (30)–(37). Stacks $\{\alpha_i\}_{i=1}^{N}$, $\{\beta_i\}_{i=1}^{N}$, $\{\gamma_i\}_{i=1}^{N}$ are called staircase-function pure strategies of the respective players.

According to Definition 1, let a discrete-time staircase-function 3-person game by time-interval breaking (19) be denoted by

$$\langle \{X(\Theta), Y(\Theta), Z(\Theta)\}, \{F(x(t), y(t), z(t)), G(x(t), y(t), z(t)), H(x(t), y(t), z(t))\} \rangle \tag{41}$$

with the players' pure strategy sets

$$X(\Theta) = \{x(t) \in X([t_1; t_2]) : x(t) = \alpha_i \in [a_{\min}; a_{\max}]\ \forall\, t \in [\tau^{(i-1)}; \tau^{(i)}) \text{ for } i = \overline{1, N-1} \text{ and } x(t) = \alpha_N \in [a_{\min}; a_{\max}]\ \forall\, t \in [\tau^{(N-1)}; \tau^{(N)}]\} \subset X([t_1; t_2]) \tag{42}$$

and

$$Y(\Theta) = \{y(t) \in Y([t_1; t_2]) : y(t) = \beta_i \in [b_{\min}; b_{\max}]\ \forall\, t \in [\tau^{(i-1)}; \tau^{(i)}) \text{ for } i = \overline{1, N-1} \text{ and } y(t) = \beta_N \in [b_{\min}; b_{\max}]\ \forall\, t \in [\tau^{(N-1)}; \tau^{(N)}]\} \subset Y([t_1; t_2]) \tag{43}$$

and

$$Z(\Theta) = \{z(t) \in Z([t_1; t_2]) : z(t) = \gamma_i \in [c_{\min}; c_{\max}]\ \forall\, t \in [\tau^{(i-1)}; \tau^{(i)}) \text{ for } i = \overline{1, N-1} \text{ and } z(t) = \gamma_N \in [c_{\min}; c_{\max}]\ \forall\, t \in [\tau^{(N-1)}; \tau^{(N)}]\} \subset Z([t_1; t_2]). \tag{44}$$

Obviously, discrete-time staircase-function 3-person game (41) is uncountably infinite as each of sets (42)–(44) contains a continuum of function-strategies. An example of situation (4) in a discrete-time staircase-function 3-person game played through 28 time units (subintervals) is given in Fig. 2, where

$$\tau^{(i)} - \tau^{(i-1)} = \frac{t_2 - t_1}{28} \quad \forall\, i = \overline{1, 28}.$$

The exemplified pure-strategy situation of three staircase functions can be also represented as a stack of 28 successive situations

$$\{\{\alpha_i, \beta_i, \gamma_i\}\}_{i=1}^{28}$$

of 28 ordinary 3-person games (39), where each ordinary pure-strategy situation

$$\{\alpha_i, \beta_i, \gamma_i\} \quad \text{for} \quad i = \overline{1, 27}$$

corresponds to a time unit (subinterval) $[\tau^{(i-1)}; \tau^{(i)})$ and ordinary pure-strategy situation

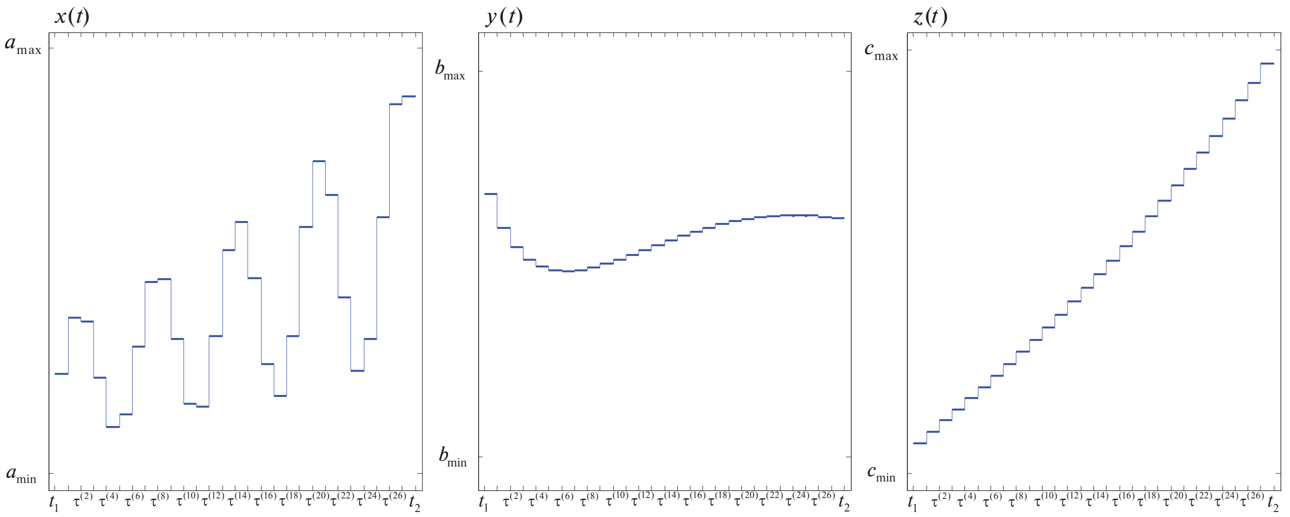


Fig. 2. A situation (4) in discrete-time staircase-function 3-person game (41), where the strategies are "digitized" versions of those strategies in Fig. 1; the game is played in uncountably infinite functional spaces (42)—(44); the exemplified pure-strategy situation is a stack of 28 successive situations $\{\{\alpha_i, \beta_i, \gamma_i\}\}_{i=1}^{28}$

$$\{\alpha_{28}, \beta_{28}, \gamma_{28}\}$$

corresponds to a time unit (subinterval)

$$\left[\tau^{(27)}; \tau^{(28)}\right] = \left[\tau^{(27)}; t_2\right].$$

Time-interval breaking (19) allows considering payoffs (5)—(7) in situation (4) equivalent to the sum of respective payoffs (32)—(37). The proof is built by an analogy to that in [14]. Another way to prove (based on 2-person games) can be found in [12].

**Theorem 1.** In a pure-strategy situation (4) of discrete-time staircase-function 3-person game (41), payoff functionals (5)—(7) are re-written as time-unit-wise sums

$$F\left(x(t), y(t), z(t)\right) = \sum_{i=1}^{N} F_i\left(\alpha_i, \beta_i, \gamma_i\right) = \sum_{i=1}^{N-1} \int\limits_{\left[\tau^{(i-1)}; \tau^{(i)}\right)} f\left(\alpha_i, \beta_i, \gamma_i, t\right) d\mu(t) + \int\limits_{\left[\tau^{(N-1)}; \tau^{(N)}\right]} f\left(\alpha_N, \beta_N, \gamma_N, t\right) d\mu(t) \quad (45)$$

and

$$G\left(x(t), y(t), z(t)\right) = \sum_{i=1}^{N} G_i\left(\alpha_i, \beta_i, \gamma_i\right) = \sum_{i=1}^{N-1} \int\limits_{\left[\tau^{(i-1)}; \tau^{(i)}\right)} g\left(\alpha_i, \beta_i, \gamma_i, t\right) d\mu(t) + \int\limits_{\left[\tau^{(N-1)}; \tau^{(N)}\right]} g\left(\alpha_N, \beta_N, \gamma_N, t\right) d\mu(t) \quad (46)$$

and

$$H\left(x(t), y(t), z(t)\right) = \sum_{i=1}^{N} H_i\left(\alpha_i, \beta_i, \gamma_i\right) = \sum_{i=1}^{N-1} \int\limits_{\left[\tau^{(i-1)}; \tau^{(i)}\right)} h\left(\alpha_i, \beta_i, \gamma_i, t\right) d\mu(t) + \int\limits_{\left[\tau^{(N-1)}; \tau^{(N)}\right]} h\left(\alpha_N, \beta_N, \gamma_N, t\right) d\mu(t), \quad (47)$$

where situation (4) is a stack of successive situations (38) in a succession of $N$ 3-person games (39).

**Proof.** Time interval $[t_1; t_2]$ can be re-written as

$$[t_1; t_2] = \left\{\bigcup_{i=1}^{N-1} \left[\tau^{(i-1)}; \tau^{(i)}\right)\right\} \cup \left[\tau^{(N-1)}; \tau^{(N)}\right]. \quad (48)$$

Therefore, the property of countable additivity of the Lebesgue integral can be used:

$$F\left(x(t), y(t), z(t)\right) = \int\limits_{[t_1; t_2]} f\left(x(t), y(t), z(t), t\right) d\mu(t) = \int\limits_{\left\{\bigcup_{i=1}^{N-1} \left[\tau^{(i-1)}; \tau^{(i)}\right)\right\} \cup \left[\tau^{(N-1)}; \tau^{(N)}\right]} f\left(x(t), y(t), z(t), t\right) d\mu(t)$$
$$= \sum_{i=1}^{N-1} \int\limits_{\left[\tau^{(i-1)}; \tau^{(i)}\right)} f\left(x(t), y(t), z(t), t\right) d\mu(t) + \int\limits_{\left[\tau^{(N-1)}; \tau^{(N)}\right]} f\left(x(t), y(t), z(t), t\right) d\mu(t). \quad (49)$$

Owing to (30), $x(t) = \alpha_i$ and $y(t) = \beta_i$ and $z(t) = \gamma_i$, so (49) is simplified as

$$\sum_{i=1}^{N-1} \int\limits_{\left[\tau^{(i-1)};\, \tau^{(i)}\right)} f\left(x(t),\, y(t),\, z(t),\, t\right) d\mu(t) + \int\limits_{\left[\tau^{(N-1)};\, \tau^{(N)}\right]} f\left(x(t),\, y(t),\, z(t),\, t\right) d\mu(t)$$
$$= \sum_{i=1}^{N-1} \int\limits_{\left[\tau^{(i-1)};\, \tau^{(i)}\right)} f\left(\alpha_i,\, \beta_i,\, \gamma_i,\, t\right) d\mu(t) + \int\limits_{\left[\tau^{(N-1)};\, \tau^{(N)}\right]} f\left(\alpha_N,\, \beta_N,\, \gamma_N,\, t\right) d\mu(t)$$
$$= \sum_{i=1}^{N} F_i\left(\alpha_i,\, \beta_i,\, \gamma_i\right). \tag{50}$$

Consequently, in discrete-time staircase-function 3-person game (41), time-unit-wise sum (45) holds in any pure-strategy situation (4) consisting of staircase-function strategies

$$x(t) \in X(\Theta), \quad y(t) \in Y(\Theta), \quad z(t) \in Z(\Theta).$$

Obviously, time-unit-wise sums (46) and (47) are proved similarly to (48)–(50). □

Theorem 1 provides a fundamental decomposition of the discrete-time staircase-function 3-person game based on the time-unit-wise summing in (45)–(47), regardless of whether the player's action space is finite or not. Although Theorem 1 itself does not provide a method of solving the game, it hints about how the game might be solved in a far easier way [12, 13, 14]. The time-unit-wise decomposition allows us to try finding an ordinary pure-strategy equilibrium in each game (39) separately, whereupon these equilibria are stitched (stacked) together [13]. Nevertheless, even a finite game (39) may not have an equilibrium in pure strategies, let alone the case when game (39) is infinite. So, in further investigation, it is presumed that every time-unit game (39) has a pure-strategy equilibrium.

A presumption about discrete-time staircase-function 3-person game (41) has an equilibrium in staircase-function pure strategies (i.e., this equilibrium is a triple of staircase functions) is equivalent to the presumption of that every time-unit game (39) has a pure-strategy equilibrium (which is a stack of time-unit equilibria). If this pure-strategy equilibrium stack is single, then every time-unit game (39) has a single pure-strategy equilibrium and vice versa [13].

**Trimatrix staircase-function game**

In a discrete-time staircase-function 3-person game (41), let the set of possible values of every player be finite. This can be done, e.g., by forcing the player to act within a finite subset of possible values of its pure strategies. Formally, the player's pure strategy set is discretized (sampled). The first player's set of possible values of its pure strategies is discretized as

$$\mathrm{A} = \left\{a_{\min} = a_i^{(0)} < a_i^{(1)} < a_i^{(2)} < \ldots < a_i^{(M-1)} < a_i^{(M)} = a_{\max}\right\} \tag{51}$$

and the second player's set of possible values of its pure strategies is discretized as

$$\mathrm{B} = \left\{b_{\min} = b_i^{(0)} < b_i^{(1)} < b_i^{(2)} < \ldots < b_i^{(Q-1)} < b_i^{(Q)} = b_{\max}\right\} \tag{52}$$

and the third player's set of possible values of its pure strategies is discretized as

$$\mathrm{C} = \left\{c_{\min} = c_i^{(0)} < c_i^{(1)} < c_i^{(2)} < \ldots < c_i^{(S-1)} < c_i^{(S)} = c_{\max}\right\} \tag{53}$$

by $M \in \mathbb{N}$ and $Q \in \mathbb{N}$ and $S \in \mathbb{N}$, where

$$a_i^{(m-1)} = a^{(m-1)} \quad \forall\, i = \overline{1,\, N} \quad \text{for} \quad m = \overline{1,\, M+1} \tag{54}$$

and

$$b_i^{(q-1)} = b^{(q-1)} \quad \forall\, i = \overline{1,\, N} \quad \text{for} \quad q = \overline{1,\, Q+1} \tag{55}$$

and

$$c_i^{(s-1)} = c^{(s-1)} \quad \forall\, i = \overline{1,\, N} \quad \text{for} \quad s = \overline{1,\, S+1}. \tag{56}$$

This means that along with the discrete time units (subintervals), the players are forced (somehow) to act within finite subsets of possible values of their pure strategies

$$A = \left\{a^{(m-1)}\right\}_{m=1}^{M+1} \tag{57}$$

and

$$B = \left\{b^{(q-1)}\right\}_{q=1}^{Q+1} \tag{58}$$

and

$$C = \left\{c^{(s-1)}\right\}_{s=1}^{S+1}. \tag{59}$$

Discretizations (51)–(56) allow defining a finite discrete-time staircase-function 3-person game, which is a trimatrix staircase-function game.

**Definition 2.** Discrete-time staircase-function 3-person game (41) is called a trimatrix staircase-function game if it is played on a product of finite subsets

$$X(\Theta, A) = \left\{x(t) \in X(\Theta) : x(t) \in \left\{a^{(m-1)}\right\}_{m=1}^{M+1}\right\} \subset X(\Theta) \subset X\left([t_1; t_2]\right) \quad (60)$$

and

$$Y(\Theta, B) = \left\{y(t) \in Y(\Theta) : y(t) \in \left\{b^{(q-1)}\right\}_{q=1}^{Q+1}\right\} \subset Y(\Theta) \subset Y\left([t_1; t_2]\right) \quad (61)$$

and

$$Z(\Theta, C) = \left\{z(t) \in Z(\Theta) : z(t) \in \left\{c^{(s-1)}\right\}_{s=1}^{S+1}\right\} \subset Z(\Theta) \subset Z\left([t_1; t_2]\right) \quad (62)$$

of sets (42)—(44). The trimatrix staircase-function game is denoted by

$$\left\langle \left\{X(\Theta, A), Y(\Theta, B), Z(\Theta, C)\right\}, \left\{F\left(x(t), y(t), z(t)\right), G\left(x(t), y(t), z(t)\right), H\left(x(t), y(t), z(t)\right)\right\}\right\rangle \quad (63)$$

by sets (42)—(44).

An example of finite sets (60)—(62) of staircase-function pure strategies in a trimatrix staircase-function game is presented in Fig. 3. The players can change their pure strategy value at most twice. Even such a pretty hard restriction grants 64 pure strategies to the first player, 27 pure strategies to the second player, and 125 pure strategies to the third player.

Obviously, the exemplified trimatrix staircase-function game can be “broken” into a succession of three ordinary trimatrix $4 \times 3 \times 5$ games, each related to its time unit (of those three units). In [13], such ordinary games were called “short”. In general, “breaking” trimatrix staircase-function game (63) into a succession of “short” games can be defined as follows.

**Definition 3.** Trimatrix staircase-function game (63) is a succession of $N$ trimatrix games can be defined as follows.

$$\left\langle \left\{\left\{a^{(m-1)}\right\}_{m=1}^{M+1}, \left\{b^{(q-1)}\right\}_{q=1}^{Q+1}, \left\{c^{(s-1)}\right\}_{s=1}^{S+1}\right\}, \left\{\mathbf{F}_i, \mathbf{G}_i, \mathbf{H}_i\right\}\right\rangle \quad \text{for} \quad i = \overline{1, N} \quad (64)$$

with the first player’s payoff matrices

$$\mathbf{F}_i = \left[\varphi_{imqs}\right]_{(M+1)\times(Q+1)\times(S+1)} \quad (65)$$

whose elements are

$$\varphi_{imqs} = \int\limits_{\left[\tau^{(i-1)};\, \tau^{(i)}\right)} f\left(a^{(m-1)}, b^{(q-1)}, c^{(s-1)}, t\right) d\mu(t) \quad \text{for} \quad i = \overline{1, N-1} \quad (66)$$

and

$$\varphi_{Nmqs} = \int\limits_{\left[\tau^{(N-1)};\, \tau^{(N)}\right]} f\left(a^{(m-1)}, b^{(q-1)}, c^{(s-1)}, t\right) d\mu(t), \quad (67)$$

with the second player’s payoff matrices

$$\mathbf{G}_i = \left[\rho_{imqs}\right]_{(M+1)\times(Q+1)\times(S+1)} \quad (68)$$

whose elements are

$$\rho_{imqs} = \int\limits_{\left[\tau^{(i-1)};\, \tau^{(i)}\right)} g\left(a^{(m-1)}, b^{(q-1)}, c^{(s-1)}, t\right) d\mu(t) \quad \text{for} \quad i = \overline{1, N-1} \quad (69)$$

and

$$\rho_{Nmqs} = \int\limits_{\left[\tau^{(N-1)};\, \tau^{(N)}\right]} g\left(a^{(m-1)}, b^{(q-1)}, c^{(s-1)}, t\right) d\mu(t), \quad (70)$$

and with the third player’s payoff matrices

$$\mathbf{H}_i = \left[\theta_{imqs}\right]_{(M+1)\times(Q+1)\times(S+1)} \quad (71)$$

whose elements are

$$\theta_{imqs} = \int\limits_{\left[\tau^{(i-1)};\, \tau^{(i)}\right)} h\left(a^{(m-1)}, b^{(q-1)}, c^{(s-1)}, t\right) d\mu(t) \quad \text{for} \quad i = \overline{1, N-1} \quad (72)$$

and

$$\theta_{Nmqs} = \int\limits_{\left[\tau^{(N-1)};\, \tau^{(N)}\right]} h\left(a^{(m-1)}, b^{(q-1)}, c^{(s-1)}, t\right) d\mu(t). \quad (73)$$

Situation (4) in the trimatrix staircase-function game is a stack of successive situations

$$\left\{\left\{a_i^{(m-1)}, b_i^{(q-1)}, c_i^{(s-1)}\right\}\right\}_{i=1}^{N} \quad (74)$$

in the succession of $N$ trimatrix games (64) by (54)—(56).

According to Definition 3, the assertion of Theorem 1 for trimatrix staircase-function game (63) can be re-written as

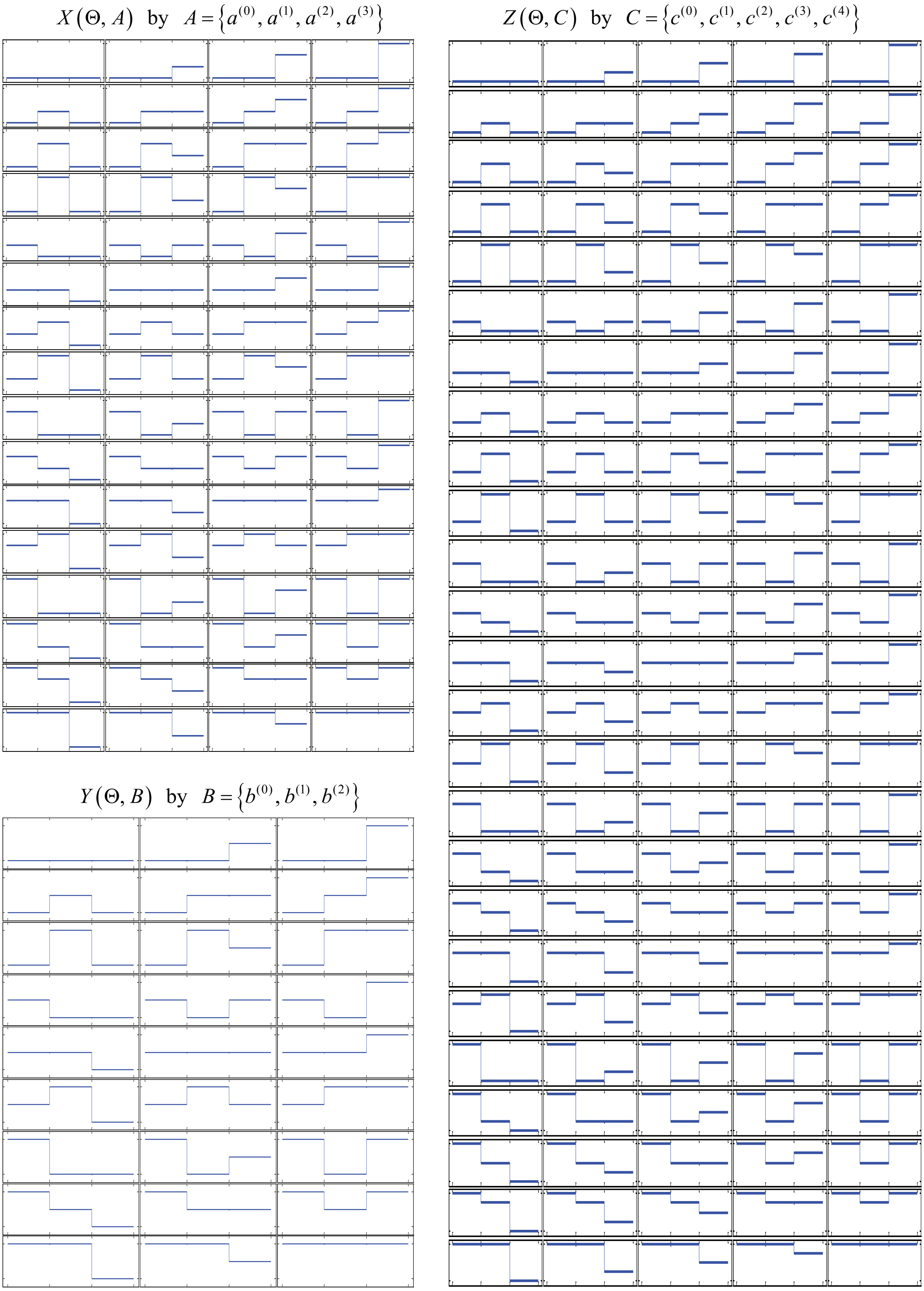


Fig. 3. An example of finite sets (60)–(62) of staircase-function pure strategies in a trimatrix staircase-function game played with 3-time-unit staircase functions of time, where the first, second, and third players have four, three, and five possible values of their pure strategies, respectively

$$F\left(x(t),\, y(t),\, z(t)\right) = \sum_{i=1}^{N} \varphi_{imqs}$$

$$= \sum_{i=1}^{N-1} \int\limits_{\left[\tau^{(i-1)};\, \tau^{(i)}\right)} f\left(a^{(m-1)},\, b^{(q-1)},\, c^{(s-1)},\, t\right) d\mu(t)$$

$$+ \int\limits_{\left[\tau^{(N-1)};\, \tau^{(N)}\right]} f\left(a^{(m-1)},\, b^{(q-1)},\, c^{(s-1)},\, t\right) d\mu(t), \quad (75)$$

$$G\left(x(t),\, y(t),\, z(t)\right) = \sum_{i=1}^{N} \rho_{imqs}$$

$$= \sum_{i=1}^{N-1} \int\limits_{\left[\tau^{(i-1)};\, \tau^{(i)}\right)} g\left(a^{(m-1)},\, b^{(q-1)},\, c^{(s-1)},\, t\right) d\mu(t)$$

$$+ \int\limits_{\left[\tau^{(N-1)};\, \tau^{(N)}\right]} g\left(a^{(m-1)},\, b^{(q-1)},\, c^{(s-1)},\, t\right) d\mu(t), \quad (76)$$

$$H\left(x(t),\, y(t),\, z(t)\right) = \sum_{i=1}^{N} \theta_{imqs}$$

$$= \sum_{i=1}^{N-1} \int\limits_{\left[\tau^{(i-1)};\, \tau^{(i)}\right)} h\left(a^{(m-1)},\, b^{(q-1)},\, c^{(s-1)},\, t\right) d\mu(t)$$

$$+ \int\limits_{\left[\tau^{(N-1)};\, \tau^{(N)}\right]} h\left(a^{(m-1)},\, b^{(q-1)},\, c^{(s-1)},\, t\right) d\mu(t). \quad (77)$$

It is worth remembering that, owing to (54)–(56), the first player's payoff in situation

$$\left\{a_i^{(m-1)},\, b_i^{(q-1)},\, c_i^{(s-1)}\right\} \quad (78)$$

is (66), (67), the second player's payoff in situation (78) is (69), (70), and the third player's payoff in situation (78) is (72), (73).

**Pure-strategy equilibrium stack**

In most practical problems, if time-unit game (39) is solved in pure strategies, there often are multiple equilibria (or even a continuum of equilibria). So, as it is presumed that every time-unit game (39) has at least a pure-strategy equilibrium, then some time-unit games may have multiple pure-strategy equilibria. The question is how to select a single equilibrium. To do this, the criterion of maximizing the sum of players' payoffs is used [12, 14, 18, 19, 23].

**Theorem 2.** If

$$\left\{\alpha_i^*,\, \beta_i^*,\, \gamma_i^*\right\} \quad (79)$$

is an equilibrium situation in 3-person game (39), $i = \overline{1,\, N}$, and maximum

$$\max_{\left\{\alpha_i^*,\, \beta_i^*,\, \gamma_i^*\right\}} \left\{F_i\left(\alpha_i^*,\, \beta_i^*,\, \gamma_i^*\right)\right.$$

$$\left. + G_i\left(\alpha_i^*,\, \beta_i^*,\, \gamma_i^*\right) + H_i\left(\alpha_i^*,\, \beta_i^*,\, \gamma_i^*\right)\right\}$$

$$= F_i\left(\alpha_i^{**},\, \beta_i^{**},\, \gamma_i^{**}\right) + G_i\left(\alpha_i^{**},\, \beta_i^{**},\, \gamma_i^{**}\right)$$

$$+ H_i\left(\alpha_i^{**},\, \beta_i^{**},\, \gamma_i^{**}\right) \quad (80)$$

is reached at an equilibrium situation

$$\left\{\alpha_i^{**},\, \beta_i^{**},\, \gamma_i^{**}\right\} \quad (81)$$

in this game for $i = \overline{1,\, N}$, then the maximum of the players' payoffs sum in a pure-strategy equilibrium stack of discrete-time staircase-function 3-person game (41) is reached at a stack

$$\left\{\left\{\alpha_i^{**},\, \beta_i^{**},\, \gamma_i^{**}\right\}\right\}_{i=1}^{N} \quad (82)$$

and this maximum is

$$p_{1,N}^{**} = \sum_{i=1}^{N} \left[F_i\left(\alpha_i^{**},\, \beta_i^{**},\, \gamma_i^{**}\right)\right.$$

$$\left. + G_i\left(\alpha_i^{**},\, \beta_i^{**},\, \gamma_i^{**}\right) + H_i\left(\alpha_i^{**},\, \beta_i^{**},\, \gamma_i^{**}\right)\right]. \quad (83)$$

**Proof.** As (81) is an equilibrium in game (39), then stack (82), owing to Theorem 4 in [13], is an equilibrium in staircase-function pure strategies in game (41). Owing to Theorem 1, the first, second, and third players' payoffs in equilibrium stack (82) are

$$u_{1,N}^{**} = \sum_{i=1}^{N} F_i\left(\alpha_i^{**},\, \beta_i^{**},\, \gamma_i^{**}\right), \quad (84)$$

$$v_{1,N}^{**} = \sum_{i=1}^{N} G_i\left(\alpha_i^{**},\, \beta_i^{**},\, \gamma_i^{**}\right), \quad (85)$$

$$w_{1,N}^{**} = \sum_{i=1}^{N} H_i\left(\alpha_i^{**},\, \beta_i^{**},\, \gamma_i^{**}\right), \quad (86)$$

respectively. Then

$$u_{1,N}^{**} + v_{1,N}^{**} + w_{1,N}^{**} = \sum_{i=1}^{N} F_i\left(\alpha_i^{**},\, \beta_i^{**},\, \gamma_i^{**}\right)$$

$$+ \sum_{i=1}^{N} G_i\left(\alpha_i^{**},\, \beta_i^{**},\, \gamma_i^{**}\right) + \sum_{i=1}^{N} H_i\left(\alpha_i^{**},\, \beta_i^{**},\, \gamma_i^{**}\right)$$

$$= \sum_{i=1}^{N} \left[F_i\left(\alpha_i^{**},\, \beta_i^{**},\, \gamma_i^{**}\right)\right.$$

$$\left. + G_i\left(\alpha_i^{**},\, \beta_i^{**},\, \gamma_i^{**}\right) + H_i\left(\alpha_i^{**},\, \beta_i^{**},\, \gamma_i^{**}\right)\right]$$

$$= \sum_{i=1}^{N} \max_{\left\{\alpha_i^*,\, \beta_i^*,\, \gamma_i^*\right\}} \left\{F_i\left(\alpha_i^*,\, \beta_i^*,\, \gamma_i^*\right)\right.$$

$$\left. + G_i\left(\alpha_i^*,\, \beta_i^*,\, \gamma_i^*\right) + H_i\left(\alpha_i^*,\, \beta_i^*,\, \gamma_i^*\right)\right\}, \quad (87)$$

where (87) is the sum of all $N$ maxima (80). □

Theorem 2 suggests a method to select the best pure-strategy equilibrium stack. Clearly, the method is correct for both discrete-time staircase-function 3-person game (41) and trimatrix staircase-function game (63). Consider now the case when a discrete-time staircase-function 3-person game is played through a lesser number of time units. Thus, instead of time-interval breaking (19), the game is played by a narrower time-interval breaking

$$\Theta_* = \{t_1 \leqslant \tau_1 = \tau^{(n)} < \tau^{(n+1)} < \tau^{(n+2)} < \ldots < \tau^{(U-1)} < \tau^{(U)} = \tau_2 \leqslant t_2\}, \tag{88}$$

where

$$n \in \left\{\overline{0,\, N-1}\right\}, \quad U \in \left\{\overline{1,\, N}\right\}, \quad n < U, \tag{89}$$

and $\left\{\tau^{(i)}\right\}_{i=n+1}^{U-1}$ are time points at which the staircase-function strategy can change its value. So, $\Theta_* \subset \Theta$ in terms of the interval breaking.

**Theorem 3.** If (79) is an equilibrium situation in 3-person game (39), $i = \overline{n+1, U}$ by (89), and maximum (80) is reached at an equilibrium situation (81) in this game for $i = \overline{n+1, U}$, then the maximum of the players' payoffs sum in a pure-strategy equilibrium stack of discrete-time staircase-function 3-person game

$$\left\langle \{X(\Theta_*),\, Y(\Theta_*),\, Z(\Theta_*)\}, \{F(x(t),\, y(t),\, z(t)),\, G(x(t),\, y(t),\, z(t)), H(x(t),\, y(t),\, z(t))\} \right\rangle \tag{90}$$

by time-interval breaking (88) is reached at a stack

$$\left\{\left\{\alpha_i^{**},\, \beta_i^{**},\, \gamma_i^{**}\right\}\right\}_{i=n+1}^{U} \tag{91}$$

and this maximum is

$$p_{n+1,U}^{**} = \sum_{i=n+1}^{U} \left[ F_i\left(\alpha_i^{**},\, \beta_i^{**},\, \gamma_i^{**}\right) + G_i\left(\alpha_i^{**},\, \beta_i^{**},\, \gamma_i^{**}\right) + H_i\left(\alpha_i^{**},\, \beta_i^{**},\, \gamma_i^{**}\right)\right]. \tag{92}$$

**Proof.** As (81) is an equilibrium in game (39), then stack (91), owing to Theorem 4 in [13], is an equilibrium in staircase-function pure strategies in game (90). Owing to Theorem 1, whose assertion remains correct for game (90) by only changing the time interval endpoints to $\tau_1 = \tau^{(n)}$ and $\tau_2 = \tau^{(U)}$, the first, second, and third players' payoffs in equilibrium stack (91) are

$$u_{n+1,U}^{**} = \sum_{i=n+1}^{U} F_i\left(\alpha_i^{**},\, \beta_i^{**},\, \gamma_i^{**}\right), \tag{93}$$

$$v_{n+1,U}^{**} = \sum_{i=n+1}^{U} G_i\left(\alpha_i^{**},\, \beta_i^{**},\, \gamma_i^{**}\right), \tag{94}$$

$$w_{n+1,U}^{**} = \sum_{i=n+1}^{U} H_i\left(\alpha_i^{**},\, \beta_i^{**},\, \gamma_i^{**}\right), \tag{95}$$

respectively. Then

$$\begin{aligned} &u_{n+1,U}^{**} + v_{n+1,U}^{**} + w_{n+1,U}^{**} \\ &= \sum_{i=n+1}^{U} F_i\left(\alpha_i^{**},\, \beta_i^{**},\, \gamma_i^{**}\right) + \sum_{i=n+1}^{U} G_i\left(\alpha_i^{**},\, \beta_i^{**},\, \gamma_i^{**}\right) + \sum_{i=n+1}^{U} H_i\left(\alpha_i^{**},\, \beta_i^{**},\, \gamma_i^{**}\right) \\ &= \sum_{i=n+1}^{U} \left[ F_i\left(\alpha_i^{**},\, \beta_i^{**},\, \gamma_i^{**}\right) + G_i\left(\alpha_i^{**},\, \beta_i^{**},\, \gamma_i^{**}\right) + H_i\left(\alpha_i^{**},\, \beta_i^{**},\, \gamma_i^{**}\right)\right] \\ &= \sum_{i=n+1}^{U} \max_{\left\{\alpha_i^*,\, \beta_i^*,\, \gamma_i^*\right\}} \left\{ F_i\left(\alpha_i^*,\, \beta_i^*,\, \gamma_i^*\right) + G_i\left(\alpha_i^*,\, \beta_i^*,\, \gamma_i^*\right) + H_i\left(\alpha_i^*,\, \beta_i^*,\, \gamma_i^*\right)\right\}, \end{aligned} \tag{96}$$

where (96) is the sum of all $U - n$ maxima (80). □

It is quite obvious that

$$\left\{\left\{\alpha_i^{**},\, \beta_i^{**},\, \gamma_i^{**}\right\}\right\}_{i=n+1}^{U} \subset \left\{\left\{\alpha_i^{**},\, \beta_i^{**},\, \gamma_i^{**}\right\}\right\}_{i=1}^{N} \tag{97}$$

regardless of whether it is a discrete-time staircase-function 3-person game (41) or a trimatrix staircase-function game (63). That is, the best pure-strategy equilibrium stack (82) in a "wider" game contains the best pure-strategy equilibrium stack (91) in a "narrower" game. Therefore, Theorem 3 along with Theorem 4 in [13] imply that the time-unit shifting does not change the structure and number of pure-strategy equilibria (it can be also a continuum on a time unit) in a discrete-time staircase-function 3-person game, nor does it change the structure of the best pure-strategy equilibrium stack determined by the maximum of the players' payoffs sum. In fact, game (90) is a subgame of discrete-time staircase-function 3-person game (41). A pure-strategy equilibrium solution of the subgame can be easily taken from the respective pure-strategy equilibrium solution (if it exists) of "wider" game (41). The best pure-strategy equilibrium stack consists of the same pure-strategy equilibria being the best for the given time units (on which the respective "short"

3-person games are played), whichever time interval encloses those time units.

**When not every "short" 3-person game is solved in pure strategies**

It is likely that, as the number of successive time units increases, there may appear at least one "short" 3-person game without any pure-strategy equilibria. So, if mixed-strategy solutions are unacceptable, then what to do in this case? Are the proved assertions and the method of selecting a single pure-strategy equilibrium still relevant then?

In fact, the existence or non-existence of pure-strategy equilibria in a time-unit 3-person game does not depend on any other time-unit game, nor does it influence the others. Therefore, an equilibrium on a time unit would not influence the equilibrium stack if the time unit was deleted (canceled, annulled, etc.). Consequently, it is sufficient to put the "wider" game on hold-up on those time units which do not have pure-strategy equilibria. In practice, it is closely equivalent to hold on while certain changes are done in the system. On the other hand, hold-ups are equivalent to nonworking days, although the latter are mostly distributed regularly (say, on weekends and holidays). Anyway, the hold-up is almost always possible to incorporate. Then time units without pure-strategy-equilibrium are just like to "disappear", and "wider" game (41) is solved, by this condition, as a pure-strategy equilibrium stack.

**Examples of 3-person games solved in staircase-function pure strategies**

Consider a finite 3-person game, in which players' payoff functionals (5)—(7) are

$$F\left(x(t),\, y(t),\, z(t)\right) = \int\limits_{[t_1;\, t_2]} \cos\left(0.8xyzt - \frac{\pi}{6}\right) e^{-0.001yzt} d\mu(t), \tag{98}$$

$$G\left(x(t),\, y(t),\, z(t)\right) = \int\limits_{[t_1;\, t_2]} \sin\left(0.6xyzt + \frac{\pi}{9}\right) d\mu(t), \tag{99}$$

$$H\left(x(t),\, y(t),\, z(t)\right) = \int\limits_{[t_1;\, t_2]} \sin\left(0.9xyzt - \frac{\pi}{8}\right) d\mu(t), \tag{100}$$

discretizations (51)—(53) are such that finite subsets (57)—(59) are

$$A = \left\{a^{(m-1)}\right\}_{m=1}^{9} = \left\{0.5 + 0.5m\right\}_{m=1}^{9} \subset [1;\, 5], \tag{101}$$

$$B = \left\{b^{(q-1)}\right\}_{q=1}^{11} = \left\{6.8 + 0.2q\right\}_{q=1}^{11} \subset [7;\, 9], \tag{102}$$

$$C = \left\{c^{(s-1)}\right\}_{s=1}^{9} = \left\{3.9 + 0.1s\right\}_{s=1}^{9} \subset [4;\, 4.8], \tag{103}$$

and the players are allowed to change their pure strategy values only at time points (the time-interval breaking is equidistant)

$$\left\{\tau^{(i)}\right\}_{i=1}^{9} = \left\{1.4\pi + 0.1\pi i\right\}_{i=1}^{9} \tag{104}$$

by $t_1 = 1.4\pi$, $t_2 = 2.4\pi$.

This finite 3-person game is a trimatrix staircase-function game being a succession of 10 trimatrix games

$$\left\langle \left\{ \left\{0.5 + 0.5m\right\}_{m=1}^{9},\, \left\{6.8 + 0.2q\right\}_{q=1}^{11},\, \left\{3.9 + 0.1s\right\}_{s=1}^{9} \right\},\, \left\{\mathbf{F}_i,\, \mathbf{G}_i,\, \mathbf{H}_i\right\} \right\rangle \tag{105}$$

with the first player's payoff $9 \times 11 \times 9$ matrices (65)

$$\mathbf{F}_i = \left[\varphi_{imqs}\right]_{9\times 11\times 9} \tag{106}$$

whose elements (66) and (67) are

$$\varphi_{imqs} = \int\limits_{[1.3\pi+0.1\pi i;\, 1.4\pi+0.1\pi i)} \cos\left(0.8\cdot\left(0.5+0.5m\right) \times\left(6.8+0.2q\right)\left(3.9+0.1s\right)t - \frac{\pi}{6}\right) \times e^{-0.001\cdot(6.8+0.2q)(3.9+0.1s)t} d\mu(t) \quad \text{for} \quad i = \overline{1,\, 9} \tag{107}$$

and

$$\varphi_{10mqs} = \int\limits_{[2.3\pi;\, 2.4\pi]} \cos\left(0.8\cdot\left(0.5+0.5m\right) \times\left(6.8+0.2q\right)\left(3.9+0.1s\right)t - \frac{\pi}{6}\right) \times e^{-0.001\cdot(6.8+0.2q)(3.9+0.1s)t} d\mu(t), \tag{108}$$

with the second player's payoff $9 \times 11 \times 9$ matrices (68)

$$\mathbf{G}_i = \left[\rho_{imqs}\right]_{9\times 11\times 9} \tag{109}$$

whose elements (69) and (70) are

$$\rho_{imqs} = \int\limits_{[1.3\pi+0.1\pi i;\, 1.4\pi+0.1\pi i)} \sin\left(0.6\cdot(0.5+0.5m) \times (6.8+0.2q)(3.9+0.1s)t+\frac{\pi}{9}\right)d\mu(t) \quad \text{for} \quad i=\overline{1, 9} \tag{110}$$

and

$$\rho_{10mqs} = \int\limits_{[2.3\pi;\, 2.4\pi]} \sin\left(0.6\cdot(0.5+0.5m) \times (6.8+0.2q)(3.9+0.1s)t+\frac{\pi}{9}\right)d\mu(t), \tag{111}$$

with the third player's payoff $9\times11\times9$ matrices (71)

$$\mathbf{H}_i = \left[\theta_{imqs}\right]_{9\times11\times9} \tag{112}$$

whose elements (72) and (73) are

$$\theta_{imqs} = \int\limits_{[1.3\pi+0.1\pi i;\, 1.4\pi+0.1\pi i)} \sin\left(0.9\cdot(0.5+0.5m) \times (6.8+0.2q)(3.9+0.1s)t-\frac{\pi}{8}\right)d\mu(t), \quad \text{for} \quad i=\overline{1, 9} \tag{113}$$

and

$$\theta_{10mqs} = \int\limits_{[2.3\pi;\, 2.4\pi]} \sin\left(0.9\cdot(0.5+0.5m) \times (6.8+0.2q)(3.9+0.1s)t-\frac{\pi}{8}\right)d\mu(t). \tag{114}$$

Each of the 10 trimatrix games (105) by (106)–(114) is solved in pure strategies. The numbers of pure-strategy equilibria on time units

$$\left\{\left\{[1.3\pi+0.1\pi i;\, 1.4\pi+0.1\pi i)\right\}_{i=1}^{9},\ [2.3\pi;\, 2.4\pi]\right\} \tag{115}$$

are 1, 2, 1, 1, 1, 1, 3, 1, 1, 1, respectively. The best pure-strategy equilibrium stack

$$\left\{\left\{\alpha_i^{**},\ \beta_i^{**},\ \gamma_i^{**}\right\}\right\}_{i=1}^{10} = \left\{\left\{a_i^{**},\ b_i^{**},\ c_i^{**}\right\}\right\}_{i=1}^{10} = \left\{x_i^{**}(t),\ y_i^{**}(t),\ z_i^{**}(t)\right\}$$

by

$$a_i^{**} \in A = \{0.5+0.5m\}_{m=1}^{9}, \quad b_i^{**} \in B = \{6.8+0.2q\}_{q=1}^{11},$$

$$c_i^{**} \in C = \{3.9+0.1s\}_{s=1}^{9},$$

at which the maximum of the players' payoffs sum $p_{1,10}^{**}$ by (83) is reached, is presented in Fig. 4, where the equilibria on time units

$$[1.5\pi;\, 1.6\pi), \quad [2\pi;\, 2.1\pi), \tag{116}$$

which do not contribute to the maximum, are shown with square-dotted line. Note that the first player's equilibrium strategies $a_2^* = 2.5$ and $a_7^* = 2.5$ not

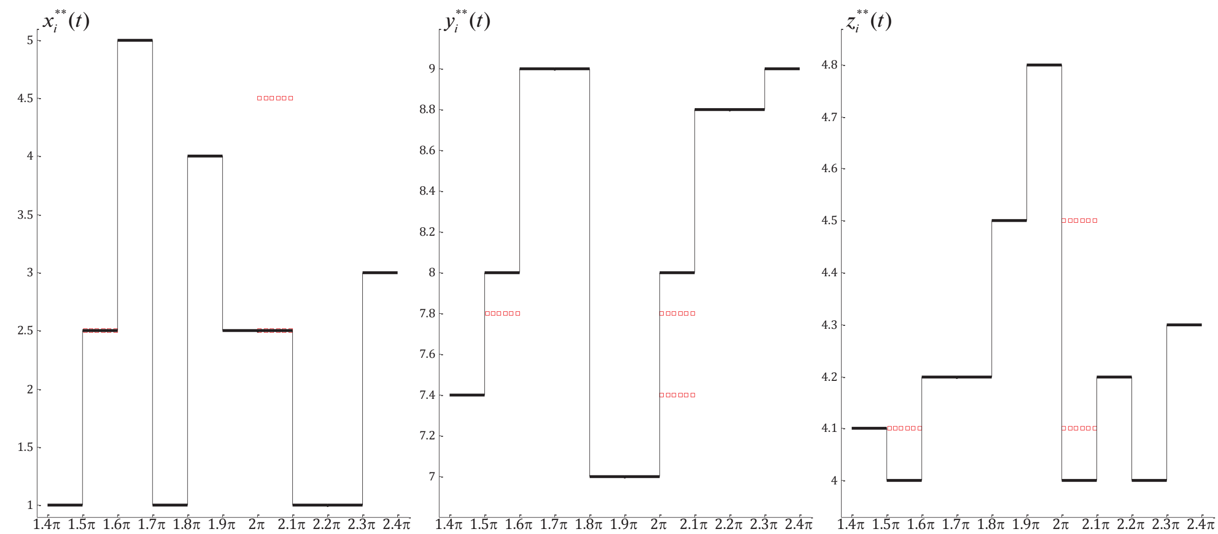


Fig. 4. The best pure-strategy equilibrium situation (as the triple of the best staircase-function pure strategy for every player) in the trimatrix staircase-function game being a succession of the 10 trimatrix games (105) by (106)–(114)

contributing to the maximum on time units (116) within the respective equilibrium situations

$$\{a_2^*, b_2^*, c_2^*\} = \{2.5, 7.8, 4.1\}$$

and

$$\{a_7^*, b_7^*, c_7^*\} = \{2.5, 7.8, 4.1\}$$

just overlap (coincide) with themselves within the best equilibrium situations

$$\{a_2^{**}, b_2^{**}, c_2^{**}\} = \{2.5, 8, 4\}$$

and

$$\{a_7^{**}, b_7^{**}, c_7^{**}\} = \{2.5, 8, 4\}$$

on (116).

Fig. 5 shows how players' payoffs

$$F_i\left(a_i^*, b_i^*, c_i^*\right) \quad \text{(shown as asterisks)},$$

$$G_i\left(a_i^*, b_i^*, c_i^*\right) \quad \text{(shown as circles)},$$

$$H_i\left(a_i^*, b_i^*, c_i^*\right) \quad \text{(shown as diamonds)}$$

in every possible equilibrium situation

$$\{a_i^*, b_i^*, c_i^*\} \quad \text{for} \quad i = \overline{1, 10} \tag{117}$$

are distributed. The best time-unit payoffs

$$F_i\left(a_i^{**}, b_i^{**}, c_i^{**}\right), \tag{118}$$

$$G_i\left(a_i^{**}, b_i^{**}, c_i^{**}\right), \tag{119}$$

$$H_i\left(a_i^{**}, b_i^{**}, c_i^{**}\right) \tag{120}$$

in every best equilibrium situation

$$\{a_i^{**}, b_i^{**}, c_i^{**}\} \quad \text{for} \quad i = \overline{1, 10} \tag{121}$$

are highlighted with squares.

Fig. 6 shows how players' payoffs

$$u_{1,k}^* = \sum_{i=1}^{k} F_i\left(a_i^*, b_i^*, c_i^*\right) \quad \text{for} \quad k = \overline{1, 10}, \tag{122}$$

$$v_{1,k}^* = \sum_{i=1}^{k} G_i\left(a_i^*, b_i^*, c_i^*\right) \quad \text{for} \quad k = \overline{1, 10}, \tag{123}$$

$$w_{1,k}^* = \sum_{i=1}^{k} H_i\left(a_i^*, b_i^*, c_i^*\right) \quad \text{for} \quad k = \overline{1, 10}, \tag{124}$$

and their best payoffs (highlighted with squares)

$$u_{1,k}^{**}, \; v_{1,k}^{**}, \; w_{1,k}^{**} \quad \text{for} \quad k = \overline{1, 10} \tag{125}$$

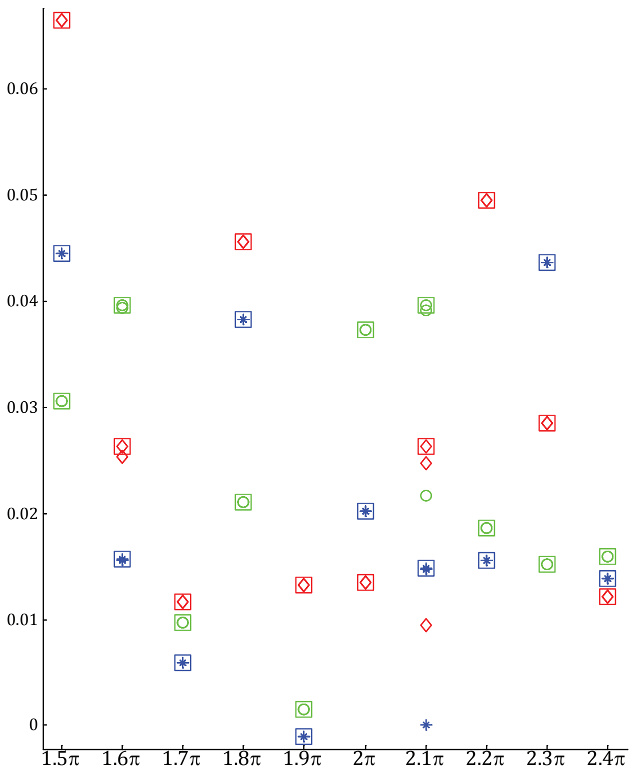


Fig. 5. Payoffs at the end of every time unit in every possible equilibrium situation (117)

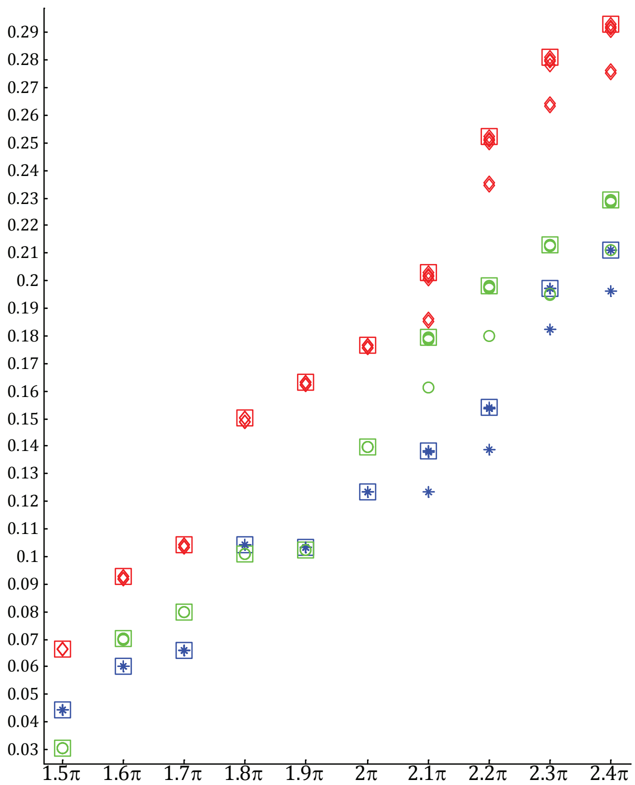


Fig. 6. Cumulative payoffs (122)–(124) and best cumulative payoffs (125) at the end of every time unit

by (84)–(86) develop as the time progresses. In fact, payoffs (122)–(125) are cumulative:

$$u_{1,1}^{*} = F_1\left(a_1^{*}, b_1^{*}, c_1^{*}\right), \quad v_{1,1}^{*} = G_1\left(a_1^{*}, b_1^{*}, c_1^{*}\right),$$
$$w_{1,1}^{*} = H_1\left(a_1^{*}, b_1^{*}, c_1^{*}\right), \quad u_{1,1}^{**} = F_1\left(a_1^{**}, b_1^{**}, c_1^{**}\right),$$
$$v_{1,1}^{**} = G_1\left(a_1^{**}, b_1^{**}, c_1^{**}\right), \quad w_{1,1}^{**} = H_1\left(a_1^{**}, b_1^{**}, c_1^{**}\right)$$

are the payoffs after the first time-unit game,

$$u_{1,10}^{*} = \sum_{i=1}^{10} F_i\left(a_i^{*}, b_i^{*}, c_i^{*}\right), \quad v_{1,10}^{*} = \sum_{i=1}^{10} G_i\left(a_i^{*}, b_i^{*}, c_i^{*}\right),$$
$$w_{1,10}^{*} = \sum_{i=1}^{10} H_i\left(a_i^{*}, b_i^{*}, c_i^{*}\right)$$

are the resulting payoffs after the end of the trimatrix staircase-function game, and

$$u_{1,10}^{**} = \sum_{i=1}^{10} F_i\left(a_i^{**}, b_i^{**}, c_i^{**}\right), \quad v_{1,10}^{**} = \sum_{i=1}^{10} G_i\left(a_i^{**}, b_i^{**}, c_i^{**}\right),$$
$$w_{1,10}^{**} = \sum_{i=1}^{10} H_i\left(a_i^{**}, b_i^{**}, c_i^{**}\right)$$

are the eventual (best) players' payoffs in the trimatrix staircase-function game being a succession of the 10 trimatrix games (105) by (106)–(114).

Fig. 7 shows how the sum of players' payoffs

$$p_{1,k}^{*} = u_{1,k}^{*} + v_{1,k}^{*} + w_{1,k}^{*} \quad \text{for} \quad k = \overline{1, 10} \tag{126}$$

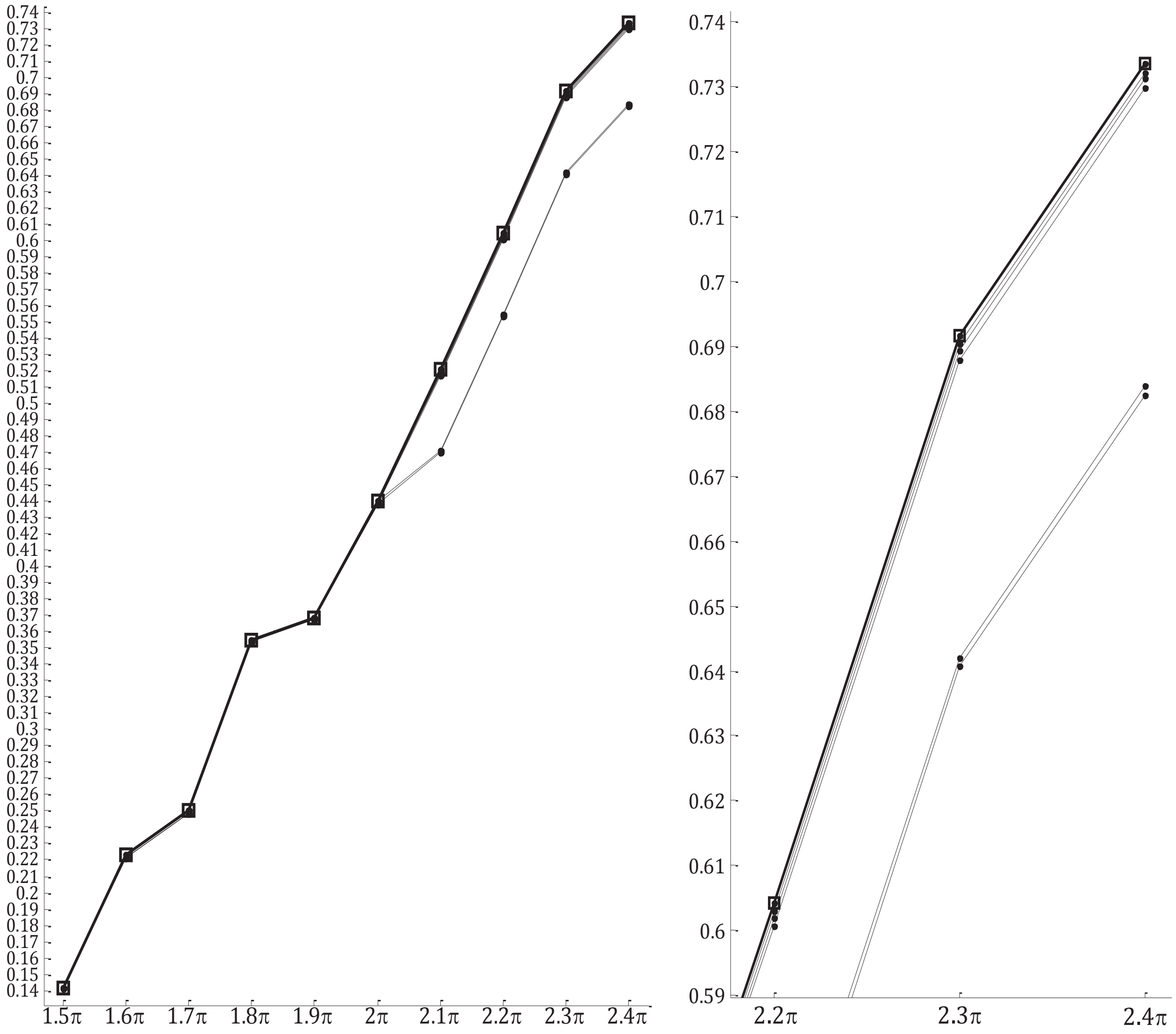


Fig. 7. Cumulative payoff sums (126) and (127) at the end of every time unit

and the best sum of players' payoffs (highlighted with thicker line and squares)

$$p_{1,k}^{**} = u_{1,k}^{**} + v_{1,k}^{**} + w_{1,k}^{**} \quad \text{for} \quad k = \overline{1, 10} \qquad (127)$$

develop as the time progresses. Due to there are six pure-strategy equilibrium stacks, there are six polylines (126), among which polyline (127) is the best (the factual payoffs sum maximum is clearly seen).

According to Theorem 3, Fig. 4 contains the best equilibrium stack in the exemplified staircase-function game defined on any subset of time units (115). In other words, if the time interval in players' payoff functionals (98)—(100) is narrowed by an integer number of time units (either from the left or right or from both endpoints), it is sufficient to narrow the time interval in Fig. 4 and extract the respective part of the best staircase-function pure strategy for every player. If only right endpoint $t_2$ is shifted to some $t = \tau_2$, then the cumulative payoffs are those at the respective time-unit end in Fig. 6 (the plot part on $(\tau_2; t_2]$ is just cut off) and their best sum is in Fig. 7 (the best-sum polyline on $(\tau_2; t_2]$ is cut off as well). If left endpoint $t_1$ is shifted to some $t = \tau_1$ (regardless of whether the right endpoint is shifted or not), the cumulative payoffs and their best sum are to be recalculated. For this, payoffs at the end of every time unit in Fig. 5 can be used.

Now, what if the exemplified staircase-function game is continued to play beyond $t_2 = 2.4\pi$? Say, when $t_1 = 2.4\pi$ and $t_2 = 20.5\pi$, and the players are still allowed to change their pure strategy values only through $0.1\pi$ time step, the staircase-function game does not have a pure-strategy equilibrium stack because there are many unit-time trimatrix games not having a pure-strategy equilibrium situation. However, putting the staircase-function game on hold-up on those time units which do not have pure-strategy equilibria allows to obtain the players' best staircase-function pure strategies with gaps (Fig. 8—10). It is up to the administrator (supervisor, manager, controller, etc.), who defines (or constrains) the rules of a system to be game-modelled, to "legalize" such gaps. Those gaps are not necessarily to be holidays or something like that. If, say, the time unit is a day, then the gap can be a day during which any activity of the players (personifying some agents on, e.g., a market) is forbidden (or suppressed) [19, 23, 27].

It is quite clear that, in real-world practice, a great deal of finite (ordinary) 3-person games do not have pure-strategy equilibria. In the case when at least one of the three players possesses an infinite set (or a continuum) of one's pure strategies, the existence of pure-strategy equilibria is far less likely. So, a "legalization" of pure-strategy solution gaps must be an additional condition imposed on the game model.

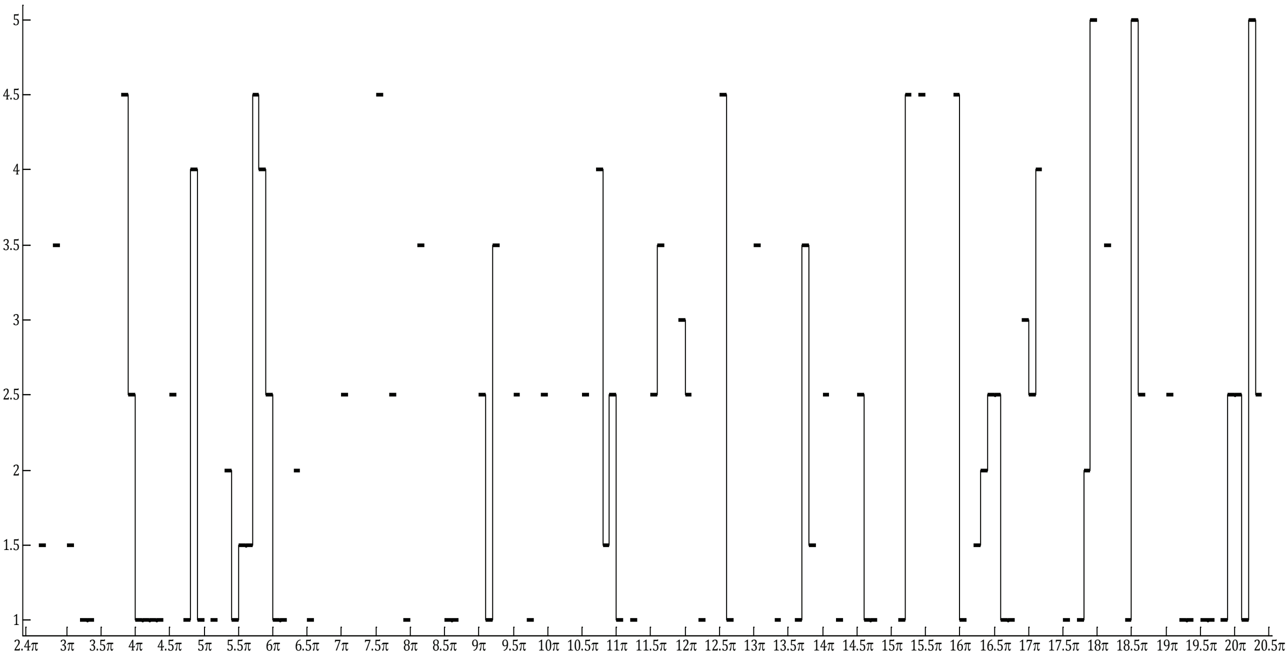


Fig. 8. The first player's best staircase-function pure strategy with gaps

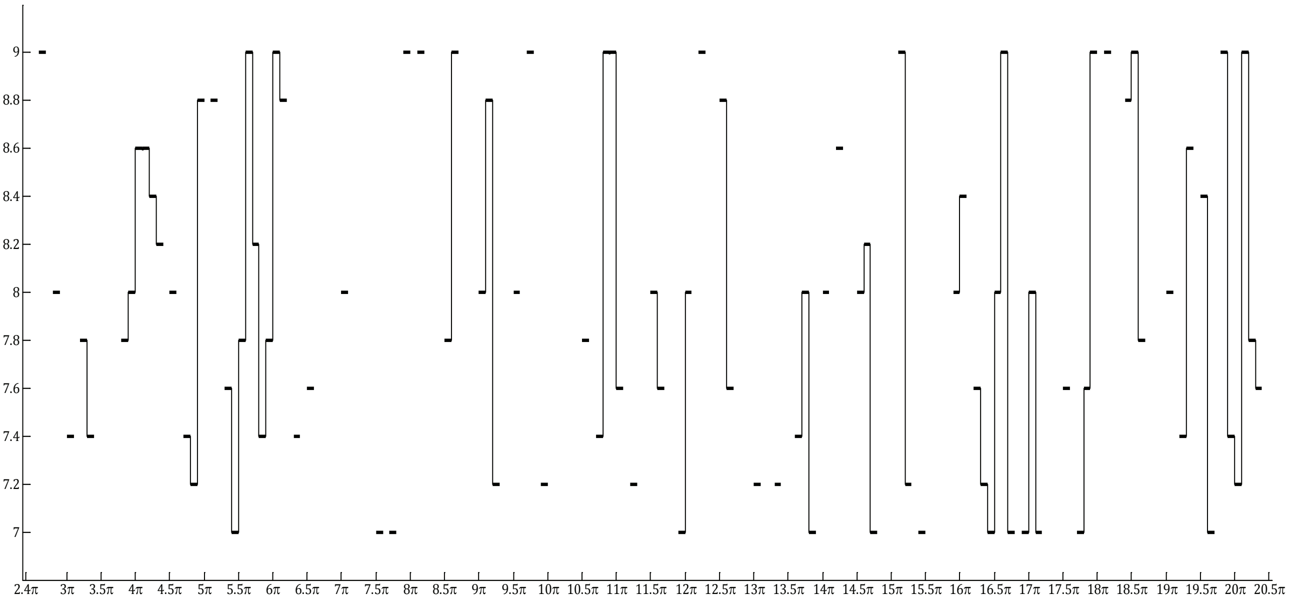

Fig. 9. The second player's best staircase-function pure strategy with gaps

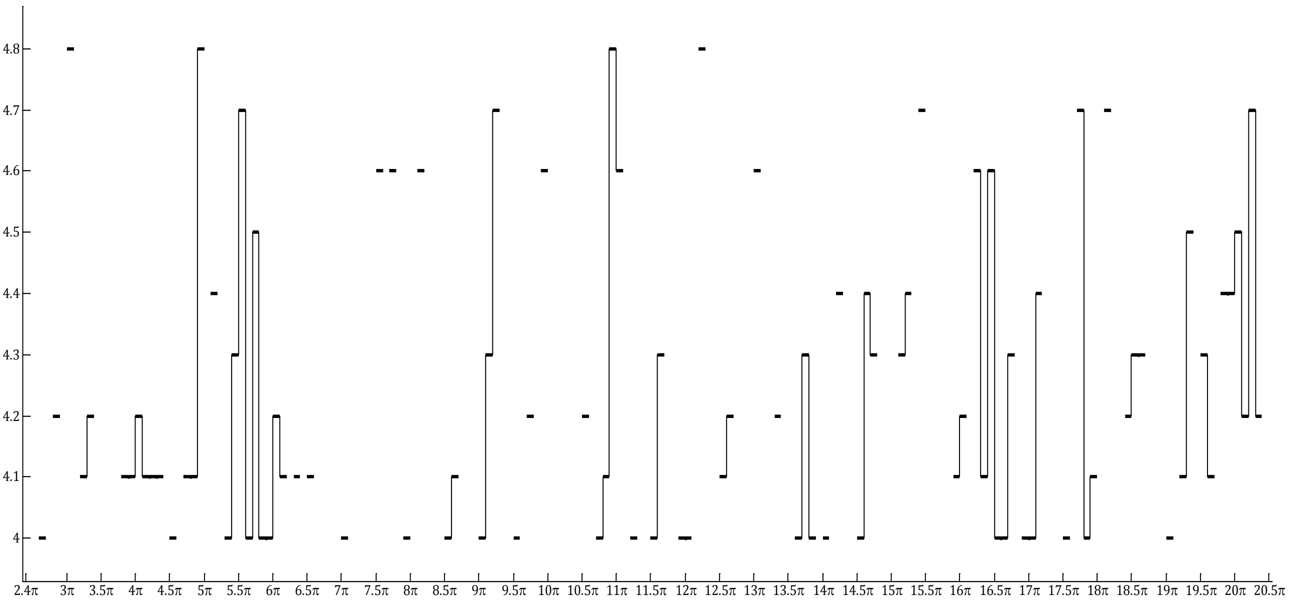

Fig. 10. The third player's best staircase-function pure strategy with gaps

**Discussion of the contribution**

The suggested method is an important supplement to the method of solving a 3-person game played in staircase-function pure strategy spaces presented in [13]. Along with the approach of the pure-strategy solution gaps, it allows quickly finding the best pure-strategy equilibrium (Theorem 2) in a discrete-time staircase-function 3-person game just by finding pure-strategy equilibria of a succession of time-unit subgames, even when not every "short" 3-person game is solved in pure strategies. In the case of a trimatrix staircase-function game, being "wider" one, its pure-strategy equilibrium situation is formed by solving and stacking pure-strategy equilibria of successive smaller-sized trimatrix games. The stacking is done in a similar manner for (uncountably) infinite games also. Then, owing to Theorem 3, the respective best equilibrium solution of any "narrower" subgame can be taken from the "wider" game best pure-strategy equilibrium. The best equilibrium situation in subgame (90) is easily found regardless of whether it is an (uncountably) infinite game or a trimatrix staircase-function game. Consequently, the suggested method is a significant contribution to the 3-person game theory and operations research, in the sense of both practical applicability and scientific soundness.

In the case of a trimatrix staircase-function game, the computational efficiency is only defined by and limited to the efficiency of finding pure-strategy equilibrium situations in an ordinary (time-unit) trimatrix game whose size is commonly not that large. Usually, this is about the direct search. Without considering the succession of time-unit trimatrix games, any straightforward approach to finding pure-strategy equilibrium situations in a trimatrix staircase-function game is intractable.

The case when the player's payoff functional has a terminal component is only seeming to be more general than that of functionals (5)–(7). Indeed, whichever terminal functions (16)–(18) are, functionals (13)–(15) can be always brought to the form of functionals (5)–(7) by transforming and fitting the terminal function under the integral. Then integrated functions (8)–(10) are respectively changed but the conception of the integral functional remains the same. This is why the terminal component case has not been considered.

Another peculiarity is the inclusion of time variable $t$ into functions (8)–(10) to be integrated. As time variable $t$ is explicitly integrated, it means that the time progress influences the process modeled by the staircase-function game. In simple terms, the explicit time variable $t$ under the integral means that something changes within the process. Contrariwise, if in a discrete-time staircase-function 3-person game time $t$ is not explicitly included in functions (8)–(10), then

$$\begin{aligned} &F_i(\alpha_i, \beta_i, \gamma_i) \\ &= \int\limits_{[\tau^{(i-1)};\, \tau^{(i)})} f(\alpha_i, \beta_i, \gamma_i)\, d\mu(t) \\ &= f(\alpha_i, \beta_i, \gamma_i) \cdot \left(\tau^{(i)} - \tau^{(i-1)}\right) \\ &\forall\, i = \overline{1, N-1} \end{aligned} \tag{128}$$

and

$$\begin{aligned} &F_N(\alpha_N, \beta_N, \gamma_N) \\ &= \int\limits_{[\tau^{(N-1)};\, \tau^{(N)}]} f(\alpha_N, \beta_N, \gamma_N)\, d\mu(t) \\ &= f(\alpha_N, \beta_N, \gamma_N) \cdot \left(\tau^{(N)} - \tau^{(N-1)}\right) \end{aligned} \tag{129}$$

instead of (32) and (33),

$$\begin{aligned} &G_i(\alpha_i, \beta_i, \gamma_i) \\ &= \int\limits_{[\tau^{(i-1)};\, \tau^{(i)})} g(\alpha_i, \beta_i, \gamma_i)\, d\mu(t) \\ &= g(\alpha_i, \beta_i, \gamma_i) \cdot \left(\tau^{(i)} - \tau^{(i-1)}\right) \\ &\forall\, i = \overline{1, N-1} \end{aligned} \tag{130}$$

and

$$\begin{aligned} &G_N(\alpha_N, \beta_N, \gamma_N) \\ &= \int\limits_{[\tau^{(N-1)};\, \tau^{(N)}]} g(\alpha_N, \beta_N, \gamma_N)\, d\mu(t) \\ &= g(\alpha_N, \beta_N, \gamma_N) \cdot \left(\tau^{(N)} - \tau^{(N-1)}\right) \end{aligned} \tag{131}$$

instead of (34) and (35), and

$$\begin{aligned} &H_i(\alpha_i, \beta_i, \gamma_i) \\ &= \int\limits_{[\tau^{(i-1)};\, \tau^{(i)})} h(\alpha_i, \beta_i, \gamma_i)\, d\mu(t) \\ &= h(\alpha_i, \beta_i, \gamma_i) \cdot \left(\tau^{(i)} - \tau^{(i-1)}\right) \\ &\forall\, i = \overline{1, N-1} \end{aligned} \tag{132}$$

and

$$\begin{aligned} &H_N(\alpha_N, \beta_N, \gamma_N) \\ &= \int\limits_{[\tau^{(N-1)};\, \tau^{(N)}]} h(\alpha_N, \beta_N, \gamma_N)\, d\mu(t) \\ &= h(\alpha_N, \beta_N, \gamma_N) \cdot \left(\tau^{(N)} - \tau^{(N-1)}\right) \end{aligned} \tag{133}$$

instead of (36) and (37). Equalities (128)–(133) mean that the player's payoff value, depending only on the time unit length, is equal to the length multiplied by the respective value of the function under the integral. If the length does not change in the case of trimatrix staircase-function game (63), then the time-unit trimatrix game does not change. If the length does not change in the case of discrete-time staircase-function 3-person game (41), the time-unit (ordinary) 3-person game defined on parallelepiped (40) does not change. Then the solution (of any type) to the initial (finite or uncountably infinite) discrete-time staircase-function 3-person game is determined just by the solution of a one time-unit game, and this solution will not change as the time units go by. Such a triviality of the equal-length-time-unit solution (by implicit time) is explained by a standstill of the players' strategies (not to be confused with equal-length-time-unit solutions, like that one in Fig. 4, where time is explicit under the integral).

Without underestimation, the scientific significance of the discrete-time staircase-function 3-person game and the methods of finding the best pure-strategy equilibrium in it (provided by Theorems 2 and 3) is high. Owing to Theorems 2 and 3, such games, if finite, are very simple models to describe struggling for rationalizing the distribution of some limited resources among three sides. Unlike an ordinary trimatrix game, which models only a static process of the struggle, a discrete-time staircase-function 3-person game considers the discrete-time dynamics of the struggle.

Searching for the best pure-strategy equilibrium is much simplified if it is somehow known that a discrete-time staircase-function 3-person game has just a single pure-strategy equilibrium situation. Then, owing to Theorem 3 in [13], every time-unit game has a single pure-strategy equilibrium. Once it is found on a time unit, the (direct) search on this unit is stopped. This is a kind of an early stop condition. It allows for significantly shortening the time of computations making thus the solving process even more efficient, especially when time-unit games are solved concurrently.

## Conclusions

Due to a pure staircase-function strategy can be considered as an ordinary mixed strategy unfolded over a time interval, staircase-function 3-person games are important mathematical objects to study ordinary ("classical") 3-person games played on a finite horizon of game rounds. Besides, staircase-function games fairly describe discrete-time dynamics of competing processes. So, building and developing a theory for staircase-function games and their solutions is an actual task and a significant contribution to the game theory field.

Directly searching for a pure-strategy equilibrium in a trimatrix staircase-function game is an intractable problem because of a gigantic size of the game rendered to an ordinary ("classical") trimatrix game. The same concerns (to a much more greater extent) a discrete-time staircase-function 3-person game, in which at least one of the three players possesses an infinite set (or a continuum) of pure staircase-function strategies. Moreover, the time interval on which the discrete-time staircase-function 3-person game, being either finite or (uncountably) infinite, is defined can vary (be shifted) by an integer number of time units. For dealing with such a time-unit shifting along with selecting a single equilibrium solution, a tractable and efficient method of finding the best pure-strategy equilibrium in a 3-person game played in finite or uncountably infinite staircase-function spaces is to solve a succession of time-unit 3-person games, whereupon their best equilibria are stacked into the best pure-strategy equilibrium. The criterion for selecting a single equilibrium solution is to maximize the players' payoffs sum. This criterion allows extracting the respective best staircase-function equilibrium pure strategy of the player in any "narrower" subgame from the player's best staircase-function equilibrium pure strategy in the "wider" game.

To deal with the case when not every time-unit 3-person game is solved in pure strategies, an effective way is to put a staircase-function game on hold-up on those time units which do not have pure-strategy equilibria. During such a "freezing" of the game, the player's payoff is not accumulated (i.e., is not added up to the preceding payoff). The players cannot change their strategies or any their activity is suppressed by attaching the respective prohibition to the game model. The result of putting the staircase-function game on hold-ups is that the player will obtain one's best staircase-function equilibrium pure strategy with gaps, whichever the time interval and time-unit shifting are.

The study might be further developed in order to consider other solution types including situations with efficient (and, maybe, non-equilibrium) strategies. Moreover, the criterion for selecting a single solution situation on each time unit can be more disputable when at least two players' payoff ranges differ significantly. Then, payoff normalization (standardization) or a correction of the criterion is to be studied as well.

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В.В. Романюк

ЗСУВ ЗА ОДИНИЦЯМИ ЧАСУ В ІГРАХ ТРЬОХ ОСІБ У СКІНЧЕННИХ І НЕЗЛІЧЕННО НЕСКІНЧЕННИХ ПРОСТОРАХ СХОДИНКОВИХ ФУНКЦІЙ, ЩО РОЗВ'ЯЗУЮТЬСЯ У ЧИСТИХ СТРАТЕГІЯХ

**Проблематика.** Ігри, котрі розігруються чистими стратегіями у формі сходинкових функцій, можуть моделювати дискретно-часову динаміку раціоналізації розподілу деяких обмежених ресурсів між гравцями. Як і ігри двох осіб, ігри трьох осіб є найбільш уживаними моделями такої раціоналізації в економіці, екології, соціальних науках, політиці, управлінні, спорті. Існує відомий метод знаходження рівноваги у грі трьох осіб, що розігрується у просторах чистих стратегій у формі сходинкових функцій. Інтервал часу, на якому така гра задається, складається із цілого числа часових одиниць. Ця рівновага утворюється укладанням рівноваг на одиницях часу. Відкритою задачею є множинність рівноваг (на деяких одиницях часу), що призводить до множинності укладів рівноваг. Ще одне відкрите питання полягає у тому, що робити із грою трьох осіб, у якій інтервал часу може бути змінений або зсунутий на ціле число часових одиниць.

**Мета дослідження.** Мета полягає у тому, щоб розвинути й удосконалити ефективний метод розв'язування ігор трьох осіб, котрі розігруються у межах скінченних множин сходинкових функцій гравців для випадку, коли період, упродовж якого гра триває, змінюється на ціле число часових одиниць.

**Методика реалізації.** Щоб досягти зазначеної мети, формалізують гру трьох осіб, в якій стратегії гравців є сходинковими функціями часу. У такій грі множина чистих стратегій гравця є континуумом сходинкових функцій. Оскільки часовий інтервал складається з часових одиниць (підінтервалів), час вважають дискретним. Після цього множина можливих значень чистої стратегії гравця дискретизується так, що гравець володіє скінченною множиною сходинкових функцій.

**Результати дослідження.** Відомий метод розвинуто так, щоб будувати єдиний уклад рівноваг у чистих стратегіях у будь-якій дискретно-часовій грі трьох осіб зі сходинковими функціями. Критерієм вибору єдиної рівноважної ситуації є максимізація суми виграшів гравців. У випадку зсуву за часовими одиницями цей критерій дозволяє витягувати відповідну найкращу рівноважну чисту стратегію у формі сходинкової функції гравця у довільній «більш вузькій» підгрі з найкращої рівноважної чистої стратегії у формі сходинкової функції цього гравця у «ширшій» грі.

**Висновки.** Ефективним методом знаходження найкращої рівноваги у чистих стратегіях у грі трьох осіб, котра розігрується у скінченних або незліченно нескінченних просторах сходинкових функцій, є розв'язування послідовності ігор трьох осіб на часових одиницях, після чого їх найкращі рівноваги укладаються у найкращу рівновагу у чистих стратегіях. У випадку, коли не кожна гра трьох осіб на часових одиницях розв'язується у чистих стратегіях, ефективним рішенням є призупинення гри зі сходинкових функцій на тих часових одиницях, котрі не мають рівноваг у чистих стратегіях. У результаті таких зупинок гравець отримуватиме власну сходинкову рівноважну чисту стратегію із пропусками, яким би не був часовий інтервал і зсув за часовими одиницями.

**Ключові слова:** теорія ігор; функціонал виграшів; гра трьох осіб; стратегія у формі сходинкової функції; триматрична гра; сходинкова рівноважна чиста стратегія.

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