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FINITE APPROXIMATION OF NON-COOPERATIVE 2-PERSON GAMES PLAYED IN STAIRCASE-FUNCTION CONTINUOUS SPACES

Background. There is a known method of approximating continuous non-cooperative 2-person games, wherein an approximate solution (an equilibrium situation) is considered acceptable if it changes minimally by changing the sampling step minimally. However, the method cannot be applied straightforwardly to a 2-person game played with staircase-function strategies. Besides, the independence of the player's sampling step selection should be taken into account.
Objective. The objective is to develop a method of finite approximation of 2-person games played in staircase-function continuous spaces by taking into account that the players are likely to independently sample their pure strategy sets.
Methods. To achieve the said objective, a 2-person game, in which the players' strategies are staircase functions of time, is formalized. In such a game, the set of the player's pure strategies is a continuum of staircase functions of time, and the time is thought of as it is discrete. The conditions of sampling the set of possible values of the player's pure strategy are stated so that the game becomes defined on a product of staircase-function finite spaces. In general, the sampling step is different at each player and the distribution of the sampled points (function-strategy values) is non-uniform.
Results. A method of finite approximation of 2-person games played in staircase-function continuous spaces is presented. The method consists in irregularly sampling the player's pure strategy value set, finding the best equilibria in "smaller" bimatrix games, each defined on a subinterval where the pure strategy value is constant, and stacking the equilibrium situations if they are consistent. The stack of the "smaller" bimatrix game equilibria is an approximate equilibrium in the initial staircase game. The (weak) consistency of the approximate equilibrium is studied by how much the payoff and equilibrium situation change as the sampling density minimally increases by the three ways of the sampling increment: only the first player's increment, only the second player's increment, both the players' increment. The consistency is decomposed into the payoff, equilibrium strategy support cardinality, equilibrium strategy sampling density, and support probability consistency. It is practically reasonable to consider a relaxed payoff consistency.
Conclusions. The suggested method of finite approximation of staircase 2-person games consists in the independent samplings, solving "smaller" bimatrix games in a reasonable time span, and stacking their solutions if they are consistent. The finite approximation is regarded appropriate if at least the respective approximate (stacked) equilibrium is ε-payoff consistent.
Keywords: game theory; payoff functional; staircase-function strategy; bimatrix game; irregular sampling; approximate equilibrium consistency.

Introduction

Non-cooperative 2-person games model processes where two sides referred to as persons or players struggle for optimizing the limited resources distribution implying as real-world resources, facilities, tools, funds, energy, etc., as well as more abstract objects whose utility is assessed as the player's payoff [1, 2]. A possible action of the player is called its (pure) strategy used to receive closely the best possible payoff under conditions of uncertainty generated by actions of the other player [3, 4]. The strategy can be as a simple (point) action, as well as a process consisting of an order of simple actions [1, 5, 6]. In the simplest case, the player's pure strategy is a short action whose duration is negligible. This negligible-duration action is represented as just a time point. In a more complicated case, the player's pure strategy is a function of time [4, 7, 8], so the player's action is a complex process [6, 9].

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Such strategies are used in multistage optimization [10], planning and control processes [11], scheduling [12], multistage corrective action processes [13], etc., modelled under uncertainties and influence of other competitive factors [5, 6, 9].

Whichever the pure strategy form is, the simplest 2-person game is a bimatrix game. Any bimatrix game has an equilibrium — a finite number or continuum of equilibria, either in pure or mixed strategies [1, 2]. Infinite or continuous 2-person games, where the players' payoff functions are meshes or surfaces of two variables defined on finite-dimensional compact Euclidean subspaces, are far more complicated [1, 2, 7, 14]. A simple example of the subspace is a unit square [2, 15]. Even if the surfaces do not have a discontinuity, the equilibrium is not always determinable as opposed to bimatrix games [2]. Moreover, 2-person games defined on open (or half-open) subspaces (e. g., open square) may not have an equilibrium at all [2, 16, 17]. Therefore, rendering a 2-person game to a bimatrix one is a crucial task in game modelling as it allows assuredly having a game solution (equilibrium point) as a pair of the players' best strategies. Without rendering, a 2-person game may have an intractable equilibrium (if any), when the equilibrium strategy support is infinite or continuous (e. g., see the examples in [1, 7, 16, 17]).

A 2-person game, in which the player's strategy is a function (e. g., of time), is a far more complicated case. In such games, the payoff kernel must be a functional mapping every pair of functions (pure strategies of the players) into a real value [7, 8, 18, 19]. A game played with such function-strategies is rendered down to a bimatrix game only when each of the players possesses a finite set of one's function-strategies. Obviously, the rendering is theoretically impossible if the set of the player's strategies is infinite.

The question of rendering an infinite game to a finite one was studied in [14, 20]. Regardless of antagonism of the players' interests, it consists in approximating the infinite game so that the approximated game would not lose the properties of the initial game. There are two fundamental conditions in the game approximation core that allow rendering a 2-person game with strategies as functions down to a bimatrix game: the time sampling and finiteness of possible values of the player's function-strategy.

According to the first fundamental condition, a time interval, on which the pure strategy is defined, should be broken into a set of subintervals, on which the strategy could be (maybe, approximately) considered constant. It can be done according to the rules of a system to be game-modelled, where the administrator (supervisor, manager, controller, etc.) does always define (or constrain) the form of the strategies players will use [1, 8, 10, 11, 13]. Moreover, any process is interpreted static on a sufficiently short time span. Henceforward, the time sampling condition is considered automatically (by default) fulfilled. Then the function-strategy becomes staircase. To keep the terminology simple, the respective game can be called staircase.

The second fundamental condition requires that the set of possible values of the player's function-strategy be finite. It is imposed for the natural reason that the number of factual actions of the players (in any game) is always finite. While the players may use strategies of whichever form they want, the number of their actions has a natural limit (unless the game is everlasting; but the everlasting game is an unreal mathematical object) [5, 7, 9, 10, 12]. Thus, the set of function-strategies used in a 2-person game is finite anyway. Therefore, any non-everlasting 2-person game is played as if it is a bimatrix game. However, the size of this bimatrix game depends on how each of the players has decided on discretizing (i. e., finitely approximating) one's set of function-strategy values. It does not seem that a player is likely to independently discretize the set identically to the other player's discretization.

A method of approximating continuous 2-person games is known from [8, 14, 20]. It is similar to the method for approximating continuous zero-sum games, but the principal difference is that there may be multiple equilibria in a 2-person game whose payoffs (unlike in a zero-sum game) are not equivalent. Theoretically, the continuous game approximation is based on sampling (discretizing) either the players' payoff kernels or the sets of players' pure strategies. Basically, this is the same as it results in finite sets of players' payoffs.

In general, an approximate solution is considered acceptable if it changes minimally by changing the sampling step minimally. This is the main requirement to accept an approximate solution. Obviously, the independence of the player's sampling step selection should be taken into consideration.

Problem statement

Although it is impossible to apply the approximation method straightforwardly to a 2-person game played with staircase-function strategies, a part of the staircase 2-person game considered on a time subinterval where the players' strategies are constant can be directly approximated by the method. Issued from the impossibility of solving 2-person games played in staircase-function continuous spaces, the objective is to develop a method of finite approximation of such games by taking into account the independence of the player's sampling step selection (i. e., the players are likely to independently sample their pure strategy sets). The approximate solution type is the Nash equilibrium. For achieving the objective, the following six tasks are to be fulfilled:

1. To formalize a non-cooperative 2-person game, in which the players' strategies are functions of time.
2. To formalize a non-cooperative 2-person game, in which the players' strategies are staircase functions. In such a game, the set of the player's pure strategies is a continuum of staircase functions of time, and the time is thought of as it is discrete.
3. To state conditions of sampling the set of possible values of the player's pure strategy so that the game be defined on a product of staircase-function finite spaces. By this, the sampling step is to be different at each player, and the distribution of the sampled points (function-strategy values) must not be necessarily uniform.
4. To state conditions of the appropriate finite approximation applicable to the non-cooperative 2-person game. This implies also the staircase-function space convergence.
5. To discuss the independence of the player's sampling step selection. The reconciliation of the difference of the players' sampling step selection on the background of multiplicity of equilibria is to be discussed as well. Eventually, the applicability and significance of the finite approximation method for the game theory is to be argued for.
6. To make an unbiased conclusion on the contribution to the game theory field. An outlook of how the research might be extended and advanced is to be made as well.

A 2-person game played with strategies as functions

Denote a pure strategy of the first and second players by $x(t)$ and $y(t)$, respectively, where each of the players uses one's strategy during (time) interval $[t_1; t_2]$ by $t_2 > t_1$. Functions $x(t)$ and $y(t)$ defined almost everywhere on interval $[t_1; t_2]$ are bounded, i. e.

$$a_{\min} \leqslant x(t) \leqslant a_{\max} \quad \text{by} \quad a_{\min} < a_{\max} \tag{1}$$

and

$$b_{\min} \leqslant y(t) \leqslant b_{\max} \quad \text{by} \quad b_{\min} < b_{\max}. \tag{2}$$

Besides, the square of the function-strategy is presumed to be Lebesgue-integrable [21]. The sets of the players' pure strategies are

$$X = \{x(t),\, t \in [t_1; t_2],\, t_1 < t_2 : a_{\min} \leqslant x(t) \leqslant a_{\max} \text{ by } a_{\min} < a_{\max}\} \subset \mathbb{L}_2[t_1; t_2] \tag{3}$$

and

$$Y = \{y(t),\, t \in [t_1; t_2],\, t_1 < t_2 : b_{\min} \leqslant y(t) \leqslant b_{\max} \text{ by } b_{\min} < b_{\max}\} \subset \mathbb{L}_2[t_1; t_2], \tag{4}$$

respectively. Each of sets (3) and (4) is a rectangular functional space, in which every element is a bounded function of time by (1) and (2).

The first player's payoff in situation

$$\{x(t),\, y(t)\} \tag{5}$$

is

$$K(x(t),\, y(t)) \tag{6}$$

and the second player's payoff in situation (5) is

$$H(x(t),\, y(t)). \tag{7}$$

Payoffs (6) and (7) are presumed to be integral functionals [21]:

$$K(x(t),\, y(t)) = \int\limits_{[t_1;\, t_2]} f(x(t),\, y(t),\, t)\, d\mu(t) \tag{8}$$

and

$$H(x(t),\, y(t)) = \int\limits_{[t_1;\, t_2]} g(x(t),\, y(t),\, t)\, d\mu(t) \tag{9}$$

with functions

$$f(x(t),\, y(t),\, t) \tag{10}$$

and

$$g(x(t),\, y(t),\, t) \tag{11}$$

of $x(t)$ and $y(t)$ explicitly including time t. Therefore, the continuous 2-person game

$$\langle \{X,\, Y\},\, \{K(x(t),\, y(t)),\, H(x(t),\, y(t))\} \rangle \tag{12}$$

is defined on product

$$X \times Y \subset \mathbb{L}_2[t_1; t_2] \times \mathbb{L}_2[t_1; t_2] \tag{13}$$

of rectangular functional spaces (3) and (4) of players' pure strategies. It is worth noting that the game continuity is defined by the continuity of spaces (3) and (4), whereas payoff functionals (8) and (9) still can have discontinuities. In general, each of payoff functionals (6) and (7) may have a terminal component like

$$K\left(x(t),\, y(t)\right) = \int_{[t_1;\, t_2]} f\left(x(t),\, y(t),\, t\right) d\mu(t) + T_f\left(x(t_2),\, y(t_2),\, t_2\right) \quad (14)$$

and

$$H\left(x(t),\, y(t)\right) = \int_{[t_1;\, t_2]} g\left(x(t),\, y(t),\, t\right) d\mu(t) + T_g\left(x(t_2),\, y(t_2),\, t_2\right) \quad (15)$$

by some terminal functions [22]

$$T_f\left(x(t_2),\, y(t_2),\, t_2\right) \quad (16)$$

and

$$T_g\left(x(t_2),\, y(t_2),\, t_2\right) \quad (17)$$

depending on only the final state of the player's strategy, but this case is not to be considered here.

A zero-sum game defined on product (13) [23] is a partial case of 2-person game (12). However, whereas the zero-sum game has an optimal solution whose payoff is constant (whichever the number of saddle points is), the 2-person game not being a zero-sum game does not have an optimal solution. It has an equilibrium point or may have multiple equilibria, at which the players' payoffs may induce contradictions with respect to payoff profitability and fairness [1, 2, 9, 24].

As it has been argued above, 2-person game (12), in which the players' strategies are functions of time, in practical reality is played discretely during time interval $[t_1;\ t_2]$. The time step is the same for each of the players because it is presumed to be established either by the rules of the system game-modelled or by the administrator. Herein, the influence of terminal functions (16), (17) is presumed to be embedded into integral functionals (8), (9).

A 2-person game with staircase-function strategies

As the 2-person game is played discretely during a time interval, then there is a number of subintervals at which the player's pure strategy is constant. Denote this number by N, where $N \in \mathbb{N} \setminus \{1\}$. Although the player's pure strategy can still have a continuum of possible values, it is now a staircase function having only N different values. So, there are $N-1$ time points at which the staircase-function strategy can change its value. These points are $\left\{\tau^{(i)}\right\}_{i=1}^{N-1}$, where

$$t_1 = \tau^{(0)} < \tau^{(1)} < \tau^{(2)} < \ldots < \tau^{(N-1)} < \tau^{(N)} = t_2. \quad (18)$$

The breaking by (18) is not necessarily to be equidistant. However, points $\left\{\tau^{(i)}\right\}_{i=0}^{N}$ are the same for each of the players. Besides, points $\left\{\tau^{(i)}\right\}_{i=0}^{N}$ do not change as the 2-person game is repeated. For real practice, surely, only a finite number of repetitions is considered (the game does not last forever).

What happens at each of those "internal" $\left\{\tau^{(i)}\right\}_{i=1}^{N-1}$ points, at which the player can "switch" the line? To answer this question, it is sufficient to imagine that the strategy value starts changing before exactly arriving at moment $t = \tau^{(i)}$. But the start should be as late as possible (that is, as close as possible to moment $t = \tau^{(i)}$). In terms of the functional analysis, this is called to be right-continuous [21, 23]. Thus, the staircase-function strategies are right-continuous: if the strategy value is changed at $t = \tau^{(i)}$, then

$$\lim_{\substack{\varepsilon>0 \\ \varepsilon\to 0}} x\left(\tau^{(i)} + \varepsilon\right) = x\left(\tau^{(i)}\right) \quad (19)$$

and

$$\lim_{\substack{\varepsilon>0 \\ \varepsilon\to 0}} y\left(\tau^{(i)} + \varepsilon\right) = y\left(\tau^{(i)}\right) \quad (20)$$

for $i = \overline{1,\, N-1}$, whereas

$$\lim_{\substack{\varepsilon>0 \\ \varepsilon\to 0}} x\left(\tau^{(i)} - \varepsilon\right) \neq x\left(\tau^{(i)}\right) \quad (21)$$

and

$$\lim_{\substack{\varepsilon>0 \\ \varepsilon\to 0}} y\left(\tau^{(i)} - \varepsilon\right) \neq y\left(\tau^{(i)}\right) \quad (22)$$

for $i=\overline{1,\,N-1}$. As an exception,

$$\lim_{\substack{\varepsilon>0\\ \varepsilon\to 0}} x\left(\tau^{(N)}-\varepsilon\right)=x\left(\tau^{(N)}\right) \tag{23}$$

and

$$\lim_{\substack{\varepsilon>0\\ \varepsilon\to 0}} y\left(\tau^{(N)}-\varepsilon\right)=y\left(\tau^{(N)}\right), \tag{24}$$

so

$$x\left(\tau^{(N-1)}\right)=x\left(\tau^{(N)}\right) \tag{25}$$

and

$$y\left(\tau^{(N-1)}\right)=y\left(\tau^{(N)}\right). \tag{26}$$

As both functions $x(t)$ and $y(t)$ are constant

$$\forall\, t\in\left[\tau^{(i-1)};\,\tau^{(i)}\right) \text{ for } i=\overline{1,\,N-1} \text{ and } \forall\, t\in\left[\tau^{(N-1)};\,\tau^{(N)}\right],$$

then game (12) can be thought of as it is a succession of N continuous 2-person games

$$\left\langle\left\{\left[a_{\min};\,a_{\max}\right],\,\left[b_{\min};\,b_{\max}\right]\right\},\,\left\{K\left(\alpha_i,\,\beta_i\right),\,H\left(\alpha_i,\,\beta_i\right)\right\}\right\rangle \tag{27}$$

defined on rectangle

$$\left[a_{\min};\,a_{\max}\right]\times\left[b_{\min};\,b_{\max}\right]$$

by

$$\alpha_i=x(t)\in\left[a_{\min};\,a_{\max}\right] \text{ and } \beta_i=y(t)\in\left[b_{\min};\,b_{\max}\right]$$
$$\forall\, t\in\left[\tau^{(i-1)};\,\tau^{(i)}\right) \text{ for } i=\overline{1,\,N-1} \text{ and } \forall\, t\in\left[\tau^{(N-1)};\,\tau^{(N)}\right], \tag{28}$$

where the factual first player's payoff in situation

$$\left\{\alpha_i,\,\beta_i\right\} \tag{29}$$

is

$$K\left(\alpha_i,\,\beta_i\right)=\int\limits_{\left[\tau^{(i-1)};\,\tau^{(i)}\right)} f\left(\alpha_i,\,\beta_i,\,t\right)d\mu(t),\quad \forall\, i=\overline{1,\,N-1} \tag{30}$$

and

$$K\left(\alpha_N,\,\beta_N\right)=\int\limits_{\left[\tau^{(N-1)};\,\tau^{(N)}\right]} f\left(\alpha_N,\,\beta_N,\,t\right)d\mu(t), \tag{31}$$

and the factual second player's payoff in situation (29) is

$$H\left(\alpha_i,\,\beta_i\right)=\int\limits_{\left[\tau^{(i-1)};\,\tau^{(i)}\right)} g\left(\alpha_i,\,\beta_i,\,t\right)d\mu(t)\quad \forall\, i=\overline{1,\,N-1} \tag{32}$$

and

$$H\left(\alpha_N,\,\beta_N\right)=\int\limits_{\left[\tau^{(N-1)};\,\tau^{(N)}\right]} g\left(\alpha_N,\,\beta_N,\,t\right)d\mu(t). \tag{33}$$

Henceforward, game (12) equivalent to the succession of N continuous 2-person games (27) by (28)—(33) is called staircase. A pure-strategy situation in staircase game (12) is a succession of N situations

$$\left\{\left\{\alpha_i,\,\beta_i\right\}\right\}_{i=1}^{N} \tag{34}$$

in games (27). In staircase game (12), the set of the player's pure strategies is still a continuum of staircase functions of time, but the time is discrete according to the breaking by (18). This time-discretization property, implying constant values of the players' strategies on every subinterval, allows, in addition to the succession of N continuous 2-person games (27), decomposing staircase game (12) with respect to the (staircase) payoff.

Theorem 1. In a pure-strategy situation (5) of staircase game (12), represented as a succession of N games (27), functionals (8) and (9) are re-written as subinterval-wise sums

$$K\left(x(t),\,y(t)\right)=\sum_{i=1}^{N}K\left(\alpha_i,\,\beta_i\right)=$$
$$=\sum_{i=1}^{N-1}\int\limits_{\left[\tau^{(i-1)};\,\tau^{(i)}\right)} f\left(\alpha_i,\,\beta_i,\,t\right)d\mu(t)+\int\limits_{\left[\tau^{(N-1)};\,\tau^{(N)}\right]} f\left(\alpha_N,\,\beta_N,\,t\right)d\mu(t) \tag{35}$$

and

$$H\left(x(t),\,y(t)\right)=\sum_{i=1}^{N}H\left(\alpha_i,\,\beta_i\right)=$$
$$=\sum_{i=1}^{N-1}\int\limits_{\left[\tau^{(i-1)};\,\tau^{(i)}\right)} g\left(\alpha_i,\,\beta_i,\,t\right)d\mu(t)+\int\limits_{\left[\tau^{(N-1)};\,\tau^{(N)}\right]} g\left(\alpha_N,\,\beta_N,\,t\right)d\mu(t), \tag{36}$$

respectively.

Proof. Time interval $[t_1; t_2]$ can be re-written as

$$[t_1; t_2] = \left\{ \bigcup_{i=1}^{N-1} \left[\tau^{(i-1)}; \tau^{(i)} \right) \right\} \cup \left[\tau^{(N-1)}; \tau^{(N)} \right]. \quad (37)$$

Therefore, the property of countable additivity of the Lebesgue integral can be used:

$$K(x(t), y(t)) = \int_{[t_1; t_2]} f(x(t), y(t), t) d\mu(t)$$

$$= \int_{\left\{ \bigcup_{i=1}^{N-1} \left[\tau^{(i-1)}; \tau^{(i)} \right) \right\} \cup \left[\tau^{(N-1)}; \tau^{(N)} \right]} f(x(t), y(t), t) d\mu(t)$$

$$= \sum_{i=1}^{N-1} \int_{\left[\tau^{(i-1)}; \tau^{(i)} \right)} f(x(t), y(t), t) d\mu(t)$$

$$+ \int_{\left[\tau^{(N-1)}; \tau^{(N)} \right]} f(x(t), y(t), t) d\mu(t). \quad (38)$$

Owing to (28), $x(t) = \alpha_i$ and $y(t) = \beta_i$, so (38) is simplified as follows:

$$\sum_{i=1}^{N-1} \int_{\left[\tau^{(i-1)}; \tau^{(i)} \right)} f(x(t), y(t), t) d\mu(t)$$

$$+ \int_{\left[\tau^{(N-1)}; \tau^{(N)} \right]} f(x(t), y(t), t) d\mu(t)$$

$$= \sum_{i=1}^{N-1} \int_{\left[\tau^{(i-1)}; \tau^{(i)} \right)} f(\alpha_i, \beta_i, t) d\mu(t)$$

$$+ \int_{\left[\tau^{(N-1)}; \tau^{(N)} \right]} f(\alpha_N, \beta_N, t) d\mu(t)$$

$$= \sum_{i=1}^{N} K(\alpha_i, \beta_i). \quad (39)$$

In staircase game (12), consequently, subinterval-wise sum (35) holds in any pure-strategy situation (5) consisting of staircase-function strategies $x(t)$ and $y(t)$. Obviously, subinterval-wise sum (36) is proved similarly to (37) — (39). □

It is noteworthy that **Theorem 1** can be proved also by considering function (10) on a subinterval as a function of time t. Denote this function by $\psi_i(t)$. Then this function appears to be zero on any other subinterval. Subsequently, function (10) is presented as the sum of those subinterval functions:

$$f(x(t), y(t), t) = \sum_{i=1}^{N} \psi_i(t),$$

whereupon (39) is deduced.

Theorem 1 does not provide a method of solving the staircase game, but it hints about how the game might be solved in an easier way. **Theorem 1** provides a fundamental decomposition of the staircase game based on the subinterval-wise summing in (35) and (36). This subinterval decomposition allows considering and solving each game (27) separately, whereupon the solutions are stitched (stacked) together.

Reasons for different and irregular sampling

Whichever game type and the number of players are, there are two main arguments for considering different sampling steps at each of the players. First, the players cannot agree on the sampling step due to the cooperation is excluded. Moreover, the players' ranges of function-strategy values may be not equal, i. e.

$$a_{\max} - a_{\min} \neq b_{\max} - b_{\min},$$

so if even the sampling step length is the same, the eventual number of the sampled points may be different. Second, if a player has a wider range of one's function-strategy values then it is likely to be sampled with a greater number of points. This, however, does not mean a denser sampling. Meanwhile, the sampling densities can be compared only when the players use strictly uniform sampling.

In general, the sampling density can vary because a player may tend to use greater or lesser values of one's function-strategy more frequently. This is a reason for a denser sampling in a neighbourhood of those values. Thus, the sampling (at least at one of the players) can be non-uniform (irregular). Therefore, in a generalized approach to finite approximation of 2-person games played in staircase-function continuous spaces, the players' samplings (along the pure strategy value axis) should be considered different and irregular. The uniform sampling will be just a partial case.

Sampling along the pure strategy value axis

In game (27) on subinterval i, the first player has its set $[a_{\min}; a_{\max}]$ of pure strategies, and the second player's pure strategy set is $[b_{\min}; b_{\max}]$. Let set $[a_{\min}; a_{\max}]$ be sampled non-uniformly (irregularly) with M points, $M \in \mathbb{N} \setminus \{1\}$:

$$A(M) = \left\{ a^{(m)} \right\}_{m=1}^{M}$$

$$= \left\{ a_{\min}, \left\{ a^{(m)} \right\}_{m=2}^{M-1}, a_{\max} \right\} \subset [a_{\min}; a_{\max}] \quad (40)$$

by

$$a^{(1)} = a_{\min} \quad \text{and} \quad a^{(M)} = a_{\max}, \tag{41}$$

i. e., the endpoints are always included into the sampling. Similarly to this, let set $[b_{\min}; b_{\max}]$ be sampled non-uniformly (irregularly) with J points, $J \in \mathbb{N} \setminus \{1\}$:

$$B(J) = \left\{b^{(j)}\right\}_{j=1}^{J} = \left\{b_{\min}, \left\{b^{(j)}\right\}_{j=2}^{J-1}, b_{\max}\right\} \subset [b_{\min}; b_{\max}] \tag{42}$$

by

$$b^{(1)} = b_{\min} \quad \text{and} \quad b^{(J)} = b_{\max}. \tag{43}$$

A pretty trivial case is the roughest sampling by $M = 2$ and $J = 2$, when

$$A(2) = \left\{a^{(1)}, a^{(2)}\right\} = \{a_{\min}, a_{\max}\} \tag{44}$$

and

$$B(2) = \left\{b^{(1)}, b^{(2)}\right\} = \{b_{\min}, b_{\max}\}, \tag{45}$$

so only the endpoints are considered without any consideration of internal points of the function-strategy value range. It is hardly possible that either of samplings (44) and (45) could be sufficient for an acceptable finite approximation, but they must be nonetheless considered for comparing them to denser samplings.

If either of integers M and J is increased by 1, a new sampling must not be of a lower density. In other words, a 1-incremented sampling must comply with the previous one. This is a requirement of the proper sampling increment.

Definition 1. Sampling

$$\Psi(S+1) = \left\{\lambda^{(s)}\right\}_{s=1}^{S+1} = \left\{\zeta_{\min}, \left\{\lambda^{(s)}\right\}_{s=2}^{S}, \zeta_{\max}\right\} \subset [\zeta_{\min}; \zeta_{\max}] \tag{46}$$

by $\zeta_{\min} < \zeta_{\max}$ and $S \in \mathbb{N} \setminus \{1\}$ is a proper sampling increment of sampling

$$\Psi(S) = \left\{\zeta^{(s)}\right\}_{s=1}^{S} = \left\{\zeta_{\min}, \left\{\zeta^{(s)}\right\}_{s=2}^{S-1}, \zeta_{\max}\right\} \subset [\zeta_{\min}; \zeta_{\max}] \tag{47}$$

if

$$\max_{s=\overline{1, S}} \left(\lambda^{(s+1)} - \lambda^{(s)}\right) < \max_{s=\overline{1, S-1}} \left(\zeta^{(s+1)} - \zeta^{(s)}\right), \tag{48}$$

i. e. the $S+1$ points in 1-incremented sampling (46) are selected denser than S points in sampling (47).

It is worth noting that the proper sampling increment does not imply the sampling density in a subrange is always increased in a 1-incremented sampling. While inequality (48) holds over the entire range between $\zeta_{\min}$ and $\zeta_{\max}$, it may not hold between a pair of neighbouring points (see **Fig. 1**).

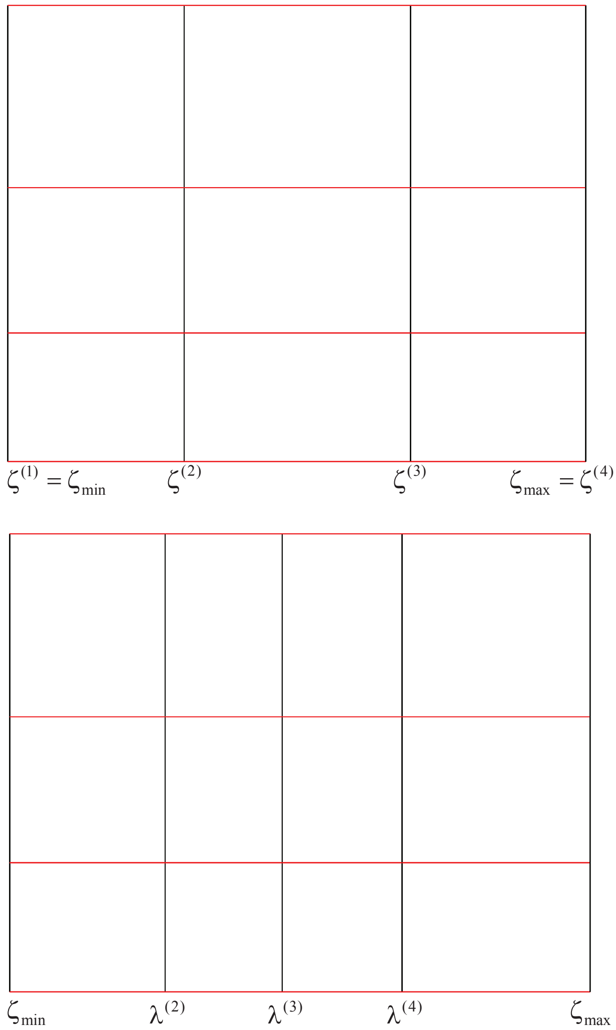

Fig. 1. A 4-point sampling (top) versus a 1-incremented sampling (bottom): although inequality (48) holds here (any subinterval length on the bottom plot is less than $\zeta^{(3)} - \zeta^{(2)}$), the right endpoint subinterval on the bottom plot has become a little bit wider ($\zeta_{\max} - \lambda^{(4)} > \zeta_{\max} - \zeta^{(3)}$)

With the sampling by (40)–(43), the succession of N continuous games (27) by (28)–(33) becomes a succession of N bimatrix $M \times J$ games

$$\left\langle \left\{ \left\{a^{(m)}\right\}_{m=1}^{M}, \left\{b^{(j)}\right\}_{j=1}^{J} \right\}, \left\{\mathbf{K}_i(M, J), \mathbf{H}_i(M, J)\right\} \right\rangle \tag{49}$$

with first player's payoff matrices

$$\mathbf{K}_i(M, J) = \left[k_{imj}(M, J)\right]_{M \times J} \tag{50}$$

whose elements are

$$k_{imj}(M, J) = \int\limits_{\left[\tau^{(i-1)};\, \tau^{(i)}\right)} f\left(a^{(m)}, b^{(j)}, t\right) d\mu(t)$$

$$\text{for} \quad i = \overline{1, N-1} \tag{51}$$

and

$$k_{Nmj}(M, J) = \int\limits_{\left[\tau^{(N-1)};\, \tau^{(N)}\right]} f\left(a^{(m)}, b^{(j)}, t\right) d\mu(t), \tag{52}$$

and with second player's payoff matrices

$$\mathbf{H}_i(M, J) = \left[h_{imj}(M, J)\right]_{M \times J} \tag{53}$$

whose elements are

$$h_{imj}(M, J) = \int\limits_{\left[\tau^{(i-1)};\, \tau^{(i)}\right)} g\left(a^{(m)}, b^{(j)}, t\right) d\mu(t)$$

$$\text{for} \quad i = \overline{1, N-1} \tag{54}$$

and

$$h_{Nmj}(M, J) = \int\limits_{\left[\tau^{(N-1)};\, \tau^{(N)}\right]} g\left(a^{(m)}, b^{(j)}, t\right) d\mu(t). \tag{55}$$

So, if integers M and J for game (12) by (28) are somehow selected, the staircase game is represented as a succession of N bimatrix $M \times J$ games (49). The representation implies that staircase game (12) and the succession of ordinary (classical) continuous 2-person games (27) are equivalent.

Definition 2. The succession of N continuous 2-person games (27) by (28)–(33) sampled by (40) and (42) is called a sampled 2-person game.

With the sampling by (40)–(43), the staircase game becomes defined on product $A(M) \times B(J)$, which becomes a product of staircase-function finite spaces by running through all $i = \overline{1, N}$. Thus, staircase game (12) becomes a finite staircase game. It might be rendered to a bimatrix game in order to obtain a staircase solution (herein, adjective "staircase" gives a hint to the type of the game, rather than to the structure of its solution). However, there is a much easier way to solve a finite staircase game.

Theorem 2. If game (12) on product (13) by conditions (1)–(11) is made a staircase game as a succession of N continuous 2-person games (27) by (28)–(33), whereupon it is sampled by (40) and (42), then the respective finite staircase game is always solved as a stack of successive equilibria of N bimatrix games (49) by (50)–(55).

Proof. An equilibrium situation in the bimatrix game always exists, either in pure or mixed strategies. Denote by

$$\mathbf{P}_i(M, J) = \left[p_i^{(m)}(M, J)\right]_{1 \times M}$$

and

$$\mathbf{Q}_i(M, J) = \left[q_i^{(j)}(M, J)\right]_{1 \times J}$$

the mixed strategies of the first and second players, respectively, in bimatrix game (49). The respective sets of mixed strategies of the first and second players are

$$\boldsymbol{\mathcal{P}} = \left\{\mathbf{P}_i(M, J) \in \mathbb{R}^M : p_i^{(m)}(M, J) \geqslant 0,\ \sum_{m=1}^{M} p_i^{(m)}(M, J) = 1\right\} \tag{56}$$

and

$$\boldsymbol{\mathcal{Q}} = \left\{\mathbf{Q}_i(M, J) \in \mathbb{R}^J : q_i^{(j)}(M, J) \geqslant 0,\ \sum_{j=1}^{J} q_i^{(j)}(M, J) = 1\right\}, \tag{57}$$

so

$$\mathbf{P}_i(M, J) \in \boldsymbol{\mathcal{P}}, \quad \mathbf{Q}_i(M, J) \in \boldsymbol{\mathcal{Q}},$$

and

$$\left\{\mathbf{P}_i(M, J), \mathbf{Q}_i(M, J)\right\} \tag{58}$$

is a situation in this game, i. e. (58) is a situation on subinterval i. Let

$$\left\{\left\{\mathbf{P}_i^*(M, J), \mathbf{Q}_i^*(M, J)\right\}\right\}_{i=1}^{N} = \left\{\left\{\left[p_i^{(m)*}(M, J)\right]_{1 \times M}, \left[q_i^{(j)*}(M, J)\right]_{1 \times J}\right\}\right\}_{i=1}^{N} \tag{59}$$

be a set of equilibria of N games (49) by (50)–(55). The stack of equilibria

$$\left\{\mathbf{P}_i^*(M, J)\right\}_{i=1}^{N} = \left\{\left[p_i^{(m)*}(M, J)\right]_{1 \times M}\right\}_{i=1}^{N} \tag{60}$$

is a stacked strategy of the first player in the staircase game (12). The stack of equilibria

$$\left\{\mathbf{Q}_i^*(M, J)\right\}_{i=1}^{N} = \left\{\left[q_i^{(j)*}(M, J)\right]_{1 \times J}\right\}_{i=1}^{N} \tag{61}$$

is a stacked strategy of the second player in the staircase game (12). Then for equilibria (59), inequalities

$$\mathbf{P}_i(M, J) \cdot \mathbf{K}_i(M, J) \cdot \left[\mathbf{Q}_i^*(M, J)\right]^{\mathrm{T}} = \sum_{m=1}^{M} \sum_{j=1}^{J} k_{imj}(M, J)\, p_i^{(m)}(M, J)\, q_i^{(j)*}(M, J)$$

$$= \sum_{m=1}^{M}\sum_{j=1}^{J} p_i^{(m)}(M, J) q_i^{(j)*}(M, J) \times \int\limits_{\left[\tau^{(i-1)};\, \tau^{(i)}\right)} f\left(a^{(m)}, b^{(j)}, t\right) d\mu(t)$$

$$\leqslant \sum_{m=1}^{M}\sum_{j=1}^{J} p_i^{(m)*}(M, J) q_i^{(j)*}(M, J) \times \int\limits_{\left[\tau^{(i-1)};\, \tau^{(i)}\right)} f\left(a^{(m)}, b^{(j)}, t\right) d\mu(t)$$

$$= \sum_{m=1}^{M}\sum_{j=1}^{J} k_{imj}(M, J) p_i^{(m)*}(M, J) q_i^{(j)*}(M, J)$$

$$= \mathbf{P}_i^*(M, J) \cdot \mathbf{K}_i(M, J) \cdot \left[\mathbf{Q}_i^*(M, J)\right]^{\mathrm{T}} = v_i^*(M, J) \quad \forall\ \mathbf{P}_i(M, J) \in \boldsymbol{P} \ \text{ for } \ i = \overline{1, N-1}, \tag{62}$$

$$\mathbf{P}_N(M, J) \cdot \mathbf{K}_N(M, J) \cdot \left[\mathbf{Q}_N^*(M, J)\right]^{\mathrm{T}}$$

$$= \sum_{m=1}^{M}\sum_{j=1}^{J} k_{Nmj}(M, J) p_N^{(m)}(M, J) q_N^{(j)*}(M, J)$$

$$= \sum_{m=1}^{M}\sum_{j=1}^{J} p_N^{(m)}(M, J) q_N^{(j)*}(M, J) \times \int\limits_{\left[\tau^{(N-1)};\, \tau^{(N)}\right]} f\left(a^{(m)}, b^{(j)}, t\right) d\mu(t)$$

$$\leqslant \sum_{m=1}^{M}\sum_{j=1}^{J} p_N^{(m)*}(M, J) q_N^{(j)*}(M, J) \times \int\limits_{\left[\tau^{(N-1)};\, \tau^{(N)}\right]} f\left(a^{(m)}, b^{(j)}, t\right) d\mu(t)$$

$$= \sum_{m=1}^{M}\sum_{j=1}^{J} k_{Nmj}(M, J) p_N^{(m)*}(M, J) q_N^{(j)*}(M, J)$$

$$= \mathbf{P}_N^*(M, J) \cdot \mathbf{K}_N(M, J) \cdot \left[\mathbf{Q}_N^*(M, J)\right]^{\mathrm{T}} = v_N^*(M, J) \quad \forall\ \mathbf{P}_N(M, J) \in \boldsymbol{P} \tag{63}$$

and inequalities

$$\mathbf{P}_i^*(M, J) \cdot \mathbf{H}_i(M, J) \cdot \left[\mathbf{Q}_i(M, J)\right]^{\mathrm{T}} =$$

$$= \sum_{m=1}^{M}\sum_{j=1}^{J} h_{imj}(M, J) p_i^{(m)*}(M, J) q_i^{(j)}(M, J)$$

$$= \sum_{m=1}^{M}\sum_{j=1}^{J} p_i^{(m)*}(M, J) q_i^{(j)}(M, J) \times \int\limits_{\left[\tau^{(i-1)};\, \tau^{(i)}\right)} g\left(a^{(m)}, b^{(j)}, t\right) d\mu(t)$$

$$\leqslant \sum_{m=1}^{M}\sum_{j=1}^{J} p_i^{(m)*}(M, J) q_i^{(j)*}(M, J) \times \int\limits_{\left[\tau^{(i-1)};\, \tau^{(i)}\right)} g\left(a^{(m)}, b^{(j)}, t\right) d\mu(t)$$

$$= \sum_{m=1}^{M}\sum_{j=1}^{J} h_{imj}(M, J) p_i^{(m)*}(M, J) q_i^{(j)*}(M, J)$$

$$= \mathbf{P}_i^*(M, J) \cdot \mathbf{H}_i(M, J) \cdot \left[\mathbf{Q}_i^*(M, J)\right]^{\mathrm{T}} = z_i^*(M, J) \quad \forall\ \mathbf{Q}_i(M, J) \in \boldsymbol{Q} \ \text{ for } \ i = \overline{1, N-1}, \tag{64}$$

$$\mathbf{P}_N^*(M, J) \cdot \mathbf{H}_N(M, J) \cdot \left[\mathbf{Q}_N(M, J)\right]^{\mathrm{T}}$$

$$= \sum_{m=1}^{M}\sum_{j=1}^{J} h_{Nmj}(M, J) p_N^{(m)*}(M, J) q_N^{(j)}(M, J)$$

$$= \sum_{m=1}^{M}\sum_{j=1}^{J} p_N^{(m)*}(M, J) q_N^{(j)}(M, J) \times \int\limits_{\left[\tau^{(N-1)};\, \tau^{(N)}\right]} g\left(a^{(m)}, b^{(j)}, t\right) d\mu(t)$$

$$\leqslant \sum_{m=1}^{M}\sum_{j=1}^{J} p_N^{(m)*}(M, J) q_N^{(j)*}(M, J) \times \int\limits_{\left[\tau^{(N-1)};\, \tau^{(N)}\right]} g\left(a^{(m)}, b^{(j)}, t\right) d\mu(t)$$

$$= \sum_{m=1}^{M}\sum_{j=1}^{J} h_{Nmj}(M, J) p_N^{(m)*}(M, J) q_N^{(j)*}(M, J)$$

$$= \mathbf{P}_N^*(M, J) \cdot \mathbf{H}_N(M, J) \cdot \left[\mathbf{Q}_N^*(M, J)\right]^{\mathrm{T}} = z_N^*(M, J) \quad \forall\ \mathbf{Q}_N(M, J) \in \boldsymbol{Q} \tag{65}$$

hold. So, inequalities

$$\sum_{i=1}^{N-1} \mathbf{P}_i(M, J) \cdot \mathbf{K}_i(M, J) \cdot \left[\mathbf{Q}_i^*(M, J)\right]^{\mathrm{T}} + \mathbf{P}_N(M, J) \cdot \mathbf{K}_N(M, J) \cdot \left[\mathbf{Q}_N^*(M, J)\right]^{\mathrm{T}}$$

$$= \sum_{i=1}^{N-1}\sum_{m=1}^{M}\sum_{j=1}^{J} k_{imj}(M, J) p_i^{(m)}(M, J) q_i^{(j)*}(M, J) + \sum_{m=1}^{M}\sum_{j=1}^{J} k_{Nmj}(M, J) p_N^{(m)}(M, J) q_N^{(j)*}(M, J)$$

$$= \sum_{i=1}^{N-1}\left(\sum_{m=1}^{M}\sum_{j=1}^{J} p_i^{(m)}(M, J) q_i^{(j)*}(M, J) \times \int\limits_{\left[\tau^{(i-1)};\, \tau^{(i)}\right)} f\left(a^{(m)}, b^{(j)}, t\right) d\mu(t)\right)$$

$$+\sum_{m=1}^{M}\sum_{j=1}^{J} p_N^{(m)}(M,J)\, q_N^{(j)*}(M,J) \times \int_{\left[\tau^{(N-1)};\,\tau^{(N)}\right]} f\left(a^{(m)}, b^{(j)}, t\right) d\mu(t)$$

$$\leqslant \sum_{i=1}^{N-1}\left(\sum_{m=1}^{M}\sum_{j=1}^{J} p_i^{(m)*}(M,J)\, q_i^{(j)*}(M,J) \times \int_{\left[\tau^{(i-1)};\,\tau^{(i)}\right)} f\left(a^{(m)}, b^{(j)}, t\right) d\mu(t)\right)$$

$$+\sum_{m=1}^{M}\sum_{j=1}^{J} p_N^{(m)*}(M,J)\, q_N^{(j)*}(M,J) \times \int_{\left[\tau^{(N-1)};\,\tau^{(N)}\right]} f\left(a^{(m)}, b^{(j)}, t\right) d\mu(t)$$

$$= \sum_{i=1}^{N-1}\sum_{m=1}^{M}\sum_{j=1}^{J} k_{imj}(M,J)\, p_i^{(m)*}(M,J)\, q_i^{(j)*}(M,J)$$

$$+\sum_{m=1}^{M}\sum_{j=1}^{J} k_{Nmj}(M,J)\, p_N^{(m)*}(M,J)\, q_N^{(j)*}(M,J)$$

$$= \sum_{i=1}^{N-1} \mathbf{P}_i^*(M,J)\cdot \mathbf{K}_i(M,J)\cdot \left[\mathbf{Q}_i^*(M,J)\right]^{\mathrm{T}}$$

$$+\mathbf{P}_N^*(M,J)\cdot \mathbf{K}_N(M,J)\cdot \left[\mathbf{Q}_N^*(M,J)\right]^{\mathrm{T}}$$

$$= \sum_{i=1}^{N} v_i^*(M,J) = v^*(M,J) \qquad (66)$$

and

$$\sum_{i=1}^{N-1} \mathbf{P}_i^*(M,J)\cdot \mathbf{H}_i(M,J)\cdot \left[\mathbf{Q}_i(M,J)\right]^{\mathrm{T}}$$

$$+\mathbf{P}_N^*(M,J)\cdot \mathbf{H}_N(M,J)\cdot \left[\mathbf{Q}_N(M,J)\right]^{\mathrm{T}}$$

$$= \sum_{i=1}^{N-1}\sum_{m=1}^{M}\sum_{j=1}^{J} h_{imj}(M,J)\, p_i^{(m)*}(M,J)\, q_i^{(j)}(M,J)$$

$$+\sum_{m=1}^{M}\sum_{j=1}^{J} h_{Nmj}(M,J)\, p_N^{(m)*}(M,J)\, q_N^{(j)}(M,J)$$

$$= \sum_{i=1}^{N-1}\left(\sum_{m=1}^{M}\sum_{j=1}^{J} p_i^{(m)*}(M,J)\, q_i^{(j)}(M,J) \times \int_{\left[\tau^{(i-1)};\,\tau^{(i)}\right)} g\left(a^{(m)}, b^{(j)}, t\right) d\mu(t)\right)$$

$$+\sum_{m=1}^{M}\sum_{j=1}^{J} p_N^{(m)*}(M,J)\, q_N^{(j)}(M,J) \times \int_{\left[\tau^{(N-1)};\,\tau^{(N)}\right]} g\left(a^{(m)}, b^{(j)}, t\right) d\mu(t)$$

$$\leqslant \sum_{i=1}^{N-1}\left(\sum_{m=1}^{M}\sum_{j=1}^{J} p_i^{(m)*}(M,J)\, q_i^{(j)*}(M,J) \times \int_{\left[\tau^{(i-1)};\,\tau^{(i)}\right)} g\left(a^{(m)}, b^{(j)}, t\right) d\mu(t)\right)$$

$$+\sum_{m=1}^{M}\sum_{j=1}^{J} p_N^{(m)*}(M,J)\, q_N^{(j)*}(M,J) \times \int_{\left[\tau^{(N-1)};\,\tau^{(N)}\right]} g\left(a^{(m)}, b^{(j)}, t\right) d\mu(t)$$

$$= \sum_{i=1}^{N-1}\sum_{m=1}^{M}\sum_{j=1}^{J} h_{imj}(M,J)\, p_i^{(m)*}(M,J)\, q_i^{(j)*}(M,J)$$

$$+\sum_{m=1}^{M}\sum_{j=1}^{J} h_{Nmj}(M,J)\, p_N^{(m)*}(M,J)\, q_N^{(j)*}(M,J)$$

$$= \sum_{i=1}^{N-1} \mathbf{P}_i^*(M,J)\cdot \mathbf{H}_i(M,J)\cdot \left[\mathbf{Q}_i^*(M,J)\right]^{\mathrm{T}}$$

$$+\mathbf{P}_N^*(M,J)\cdot \mathbf{H}_N(M,J)\cdot \left[\mathbf{Q}_N^*(M,J)\right]^{\mathrm{T}}$$

$$= \sum_{i=1}^{N} z_i^*(M,J) = z^*(M,J) \qquad (67)$$

hold as well. Therefore, inequalities (66) and (67) along with using **Theorem 1** allow concluding that the stack of successive equilibria (59) is an equilibrium in game (12) by (27) sampled by (40), (42). □

It is quite clear that the solutions of the $M \times J$ bimatrix games are independent. So these $M \times J$ bimatrix games can be solved in parallel, without caring of the succession. The succession does matter when the solutions are stacked (stitched) together to form the staircase solution (the solution to the finite staircase game). Once N equilibria in the ("smaller" or "short") bimatrix games are found, they are successively stacked and the stack, according to **Theorem 2,** is an equilibrium in the staircase game (12) sampled by (40), (42).

A corollary of **Theorem 2** is that any combination of the respective equilibria of the "short" bimatrix games is an equilibrium of the sampled 2-person game. Multiplicity of equilibria on a subinterval tied to multiplicity of equilibria on other subintervals leads to a sudden growth of the stacked equilibria (the stack of the N successive equilibria). Besides, there often happen bimatrix games with a continuum of equilibria (e. g., the continuum is constituted by a linear combination of two equilibrium points). This problem makes a fundamental difference between approximating a zero-sum staircase game and a 2-person staircase game (which is not a zero-sum one).

If all N bimatrix games are solved in pure strategies, then stacking the equilibria is fulfilled trivially. When there is at least an equilibrium in mixed strategies for a subinterval, the stacking is fulfilled as well implying that the resulting pure-mixed-strategy equilibrium of staircase game (12) is realized successively, subinterval by subinterval, spending the same amount of time to implement both pure strategy and mixed strategy equilibria (e. g., see [2, 7, 9, 12, 14]). Nevertheless, stacking up pure-strategy equilibria and mixed-strategy equilibria of $M \times J$ bimatrix games (49) can be cumbersome. The best case is when every "short" game has a single pure-strategy equilibrium.

Consistency of approximate equilibrium

In the case of the non-cooperative 2-person game, the conditions of the appropriate finite approximation are stated by using the known method of approximating isomorphic infinite 2-person non-cooperative games via variously sampling the players' payoff functions and reshaping payoff matrices into bimatrix game [20]. The method uses uniform sampling, but it is easy to generalize it. There are five items of the conditions. The requirement of the smooth sampling of the payoff kernel is inapplicable here [24].

First of all, there is an easy-to-find condition of the finite approximation appropriateness. It is about the equilibrium payoff change, which must not change more by the proper sampling increment. Inasmuch as an increment is possible from the side of both the players, then this condition is a set of $6N$ inequalities:

$$\left|v_i^*(M, J) - v_i^*(M+1, J)\right| \leqslant \left|v_i^*(M-1, J) - v_i^*(M, J)\right| \quad \text{for} \quad i = \overline{1, N}, \tag{68}$$

$$\left|z_i^*(M, J) - z_i^*(M+1, J)\right| \leqslant \left|z_i^*(M-1, J) - z_i^*(M, J)\right| \quad \text{for} \quad i = \overline{1, N}, \tag{69}$$

$$\left|v_i^*(M, J) - v_i^*(M, J+1)\right| \leqslant \left|v_i^*(M, J-1) - v_i^*(M, J)\right| \quad \text{for} \quad i = \overline{1, N}, \tag{70}$$

$$\left|z_i^*(M, J) - z_i^*(M, J+1)\right| \leqslant \left|z_i^*(M, J-1) - z_i^*(M, J)\right| \quad \text{for} \quad i = \overline{1, N}, \tag{71}$$

$$\left|v_i^*(M, J) - v_i^*(M+1, J+1)\right| \leqslant \left|v_i^*(M-1, J-1) - v_i^*(M, J)\right| \quad \text{for} \quad i = \overline{1, N}, \tag{72}$$

$$\left|z_i^*(M, J) - z_i^*(M+1, J+1)\right| \leqslant \left|z_i^*(M-1, J-1) - z_i^*(M, J)\right| \quad \text{for} \quad i = \overline{1, N}. \tag{73}$$

Conditions (68)—(73) mean that, as the sampling density minimally increases, either from the side of the first or second player (or both), an equilibrium payoff change for both the first and second players in an appropriate approximation should not grow.

Definition 3. An approximate equilibrium (59) in staircase game (12) is called payoff-{M, J}-consistent if inequalities (68)—(73) hold. Stack (60) is called first-player-payoff-{M, J}-consistent if inequalities (68), (70), (72) hold. Stack (61) is called second-player-payoff-{M, J}-consistent if inequalities (69), (71), (73) hold.

The second condition is the change of the equilibrium strategy support cardinality. Denote the supports of the equilibrium strategies of the players by

$$\operatorname{supp} \mathbf{P}_i^*(M, J) = \{m_u\}_{u=1}^{U_i(M, J)} \subset \{m\}_{m=1}^{M} \tag{74}$$

by the respective support probabilities

$$\left\{p_i^{(m_u)*}(M, J)\right\}_{u=1}^{U_i(M, J)} \tag{75}$$

and

$$\operatorname{supp} \mathbf{Q}_i^*(M, J) = \{j_w\}_{w=1}^{W_i(M, J)} \subset \{j\}_{j=1}^{J} \tag{76}$$

by the respective support probabilities

$$\left\{q_i^{(j_w)*}(M, J)\right\}_{w=1}^{W_i(M, J)}. \tag{77}$$

Then $6N$ inequalities

$$U_i(M+1, J) \geqslant U_i(M, J) \quad \text{for} \quad i = \overline{1, N}, \tag{78}$$

$$U_i(M, J+1) \geqslant U_i(M, J) \quad \text{for} \quad i = \overline{1, N}, \tag{79}$$

$$U_i(M+1, J+1) \geqslant U_i(M, J) \quad \text{for} \quad i = \overline{1, N}, \tag{80}$$

$$W_i(M+1, J) \geqslant W_i(M, J) \quad \text{for} \quad i = \overline{1, N}, \tag{81}$$

$$W_i(M, J+1) \geqslant W_i(M, J) \quad \text{for} \quad i = \overline{1, N}, \tag{82}$$

$$W_i(M+1, J+1) \geqslant W_i(M, J) \quad \text{for} \quad i = \overline{1, N} \tag{83}$$

require that, by minimally increasing the sampling density, either from the side of the first or second player (or both), the cardinalities of the supports not decrease.

Definition 4. An approximate equilibrium (59) in staircase game (12) is called weakly support-cardinality-$\{M, J\}$-consistent if inequalities (78)—(83) hold. Support (74) is called weakly first-player-support-cardinality-$\{M, J\}$-consistent if inequalities (78)—(80) hold. Support (76) is called weakly second-player-support-cardinality-$\{M, J\}$-consistent if inequalities (81)—(83) hold.

Obviously, requirements (78)—(83) can be supplemented (strengthened) by considering a minimal decrement of the sampling density. Then another $6N$ inequalities

$$U_i(M, J) \geqslant U_i(M-1, J) \quad \text{for} \quad i = \overline{1, N}, \tag{84}$$

$$U_i(M, J) \geqslant U_i(M, J-1) \quad \text{for} \quad i = \overline{1, N}, \tag{85}$$

$$U_i(M, J) \geqslant U_i(M-1, J-1) \quad \text{for} \quad i = \overline{1, N}, \tag{86}$$

$$W_i(M, J) \geqslant W_i(M-1, J) \quad \text{for} \quad i = \overline{1, N}, \tag{87}$$

$$W_i(M, J) \geqslant W_i(M, J-1) \quad \text{for} \quad i = \overline{1, N}, \tag{88}$$

$$W_i(M, J) \geqslant W_i(M-1, J-1) \quad \text{for} \quad i = \overline{1, N} \tag{89}$$

are required.

Definition 5. An approximate equilibrium (59) in staircase game (12) is called support-cardinality-$\{M, J\}$-consistent if inequalities (78)—(89) hold. Support (74) is called first-player-support-cardinality-$\{M, J\}$-consistent if inequalities (78)—(80) and (84)—(86) hold. Support (76) is called second-player-support-cardinality-$\{M, J\}$-consistent if inequalities (81)—(83) and (87)—(89) hold.

As the sampling density minimally increases, the maximal gap between the support indices should not increase. Let $m_u(M, J)$ and $j_w(M, J)$ be the respective support indices corresponding to integers $\{M, J\}$ on a subinterval by (28). Then $6N$ inequalities

$$\max_{u=\overline{1, U_i(M+1, J)-1}} \left[m_{u+1}(M+1, J) - m_u(M+1, J) \right] \leqslant \max_{u=\overline{1, U_i(M, J)-1}} \left[m_{u+1}(M, J) - m_u(M, J) \right] \quad \text{for} \quad i = \overline{1, N}, \tag{90}$$

$$\max_{u=\overline{1, U_i(M, J+1)-1}} \left[m_{u+1}(M, J+1) - m_u(M, J+1) \right] \leqslant \max_{u=\overline{1, U_i(M, J)-1}} \left[m_{u+1}(M, J) - m_u(M, J) \right] \quad \text{for} \quad i = \overline{1, N}, \tag{91}$$

$$\max_{u=\overline{1, U_i(M+1, J+1)-1}} \times \left[m_{u+1}(M+1, J+1) - m_u(M+1, J+1) \right] \leqslant \max_{u=\overline{1, U_i(M, J)-1}} \left[m_{u+1}(M, J) - m_u(M, J) \right] \quad \text{for} \quad i = \overline{1, N}, \tag{92}$$

$$\max_{w=\overline{1, W_i(M+1, J)-1}} \left[j_{w+1}(M+1, J) - j_w(M+1, J) \right] \leqslant \max_{w=\overline{1, W_i(M, J)-1}} \left[j_{w+1}(M, J) - j_w(M, J) \right] \quad \text{for} \quad i = \overline{1, N}, \tag{93}$$

$$\max_{w=\overline{1, W_i(M, J+1)-1}} \left[j_{w+1}(M, J+1) - j_w(M, J+1) \right] \leqslant \max_{w=\overline{1, W_i(M, J)-1}} \left[j_{w+1}(M, J) - j_w(M, J) \right] \quad \text{for} \quad i = \overline{1, N}, \tag{94}$$

$$\max_{w=\overline{1, W_i(M+1, J+1)-1}} \times \left[j_{w+1}(M+1, J+1) - j_w(M+1, J+1) \right] \leqslant \max_{w=\overline{1, W_i(M, J)-1}} \left[j_{w+1}(M, J) - j_w(M, J) \right] \quad \text{for} \quad i = \overline{1, N} \tag{95}$$

are required.

Definition 6. An approximate equilibrium (59) in staircase game (12) is called weakly sampling-density-$\{M, J\}$-consistent if inequalities (90)—(95) hold. Support (74) is called weakly first-player-sampling-density-$\{M, J\}$-consistent if inequalities (90)—(92) hold. Support (76) is called weakly second-player-sampling-density-$\{M, J\}$-consistent if inequalities (93)—(95) hold.

Similarly to strengthening the weak (by **Definition 4**) support cardinality to that by **Definition 5**, requirements (90)—(95) can be strengthened by considering a minimal decrement of the sampling density. Then another $6N$ inequalities

$$\max_{u=\overline{1, U_i(M, J)-1}} \left[m_{u+1}(M, J) - m_u(M, J) \right] \leqslant \max_{u=\overline{1, U_i(M-1, J)-1}} \left[m_{u+1}(M-1, J) - m_u(M-1, J) \right] \quad \text{for} \quad i = \overline{1, N}, \tag{96}$$

$$\max_{u=\overline{1,\, U_i(M,\,J)-1}} \left[m_{u+1}(M, J) - m_u(M, J) \right] \leqslant \max_{u=\overline{1,\, U_i(M,\,J-1)-1}} \left[m_{u+1}(M, J-1) - m_u(M, J-1) \right] \quad \text{for} \quad i = \overline{1, N}, \tag{97}$$

$$\max_{u=\overline{1,\, U_i(M,\,J)-1}} \left[m_{u+1}(M, J) - m_u(M, J) \right] \leqslant \max_{u=\overline{1,\, U_i(M-1,\,J-1)-1}} \times \left[m_{u+1}(M-1, J-1) - m_u(M-1, J-1) \right] \quad \text{for} \quad i = \overline{1, N}, \tag{98}$$

$$\max_{w=\overline{1,\, W_i(M,\,J)-1}} \left[j_{w+1}(M, J) - j_w(M, J) \right] \leqslant \max_{w=\overline{1,\, W_i(M-1,\,J)-1}} \left[j_{w+1}(M-1, J) - j_w(M-1, J) \right] \quad \text{for} \quad i = \overline{1, N}, \tag{99}$$

$$\max_{w=\overline{1,\, W_i(M,\,J)-1}} \left[j_{w+1}(M, J) - j_w(M, J) \right] \leqslant \max_{w=\overline{1,\, W_i(M,\,J-1)-1}} \left[j_{w+1}(M, J-1) - j_w(M, J-1) \right] \quad \text{for} \quad i = \overline{1, N}, \tag{100}$$

$$\max_{w=\overline{1,\, W_i(M,\,J)-1}} \left[j_{w+1}(M, J) - j_w(M, J) \right] \leqslant \max_{w=\overline{1,\, W_i(M-1,\,J-1)-1}} \times \left[j_{w+1}(M-1, J-1) - j_w(M-1, J-1) \right] \quad \text{for} \quad i = \overline{1, N} \tag{101}$$

are required.

Definition 7. An approximate equilibrium (59) in staircase game (12) is called sampling-density-$\{M, J\}$-consistent if inequalities (90)–(101) hold. Support (74) is called first-player-sampling-density-$\{M, J\}$-consistent if inequalities (90)–(92) and (96)–(98) hold. Support (76) is called second-player-sampling-density-$\{M, J\}$-consistent if inequalities (93)–(95) and (99)–(101) hold.

Denote by $h_1(i;\, m,\, M,\, J)$ a polyline whose vertices are probabilities

$$\left\{ p_i^{(m)*}(M, J) \right\}_{m=1}^{M},$$

and denote by $h_2(i;\, j,\, M,\, J)$ a polyline whose vertices are probabilities

$$\left\{ q_i^{(j)*}(M, J) \right\}_{j=1}^{J}.$$

Then, by minimally increasing the sampling density, the "neighbouring" polylines should not be farther from each other, i. e. inequalities

$$\max_{[0;\,1]} \left| h_1(i;\, m,\, M,\, J) - h_1(i;\, m,\, M+1,\, J) \right| \leqslant \max_{[0;\,1]} \left| h_1(i;\, m,\, M-1,\, J) - h_1(i;\, m,\, M,\, J) \right| \quad \text{for} \quad i = \overline{1, N}, \tag{102}$$

$$\max_{[0;\,1]} \left| h_1(i;\, m,\, M,\, J) - h_1(i;\, m,\, M,\, J+1) \right| \leqslant \max_{[0;\,1]} \left| h_1(i;\, m,\, M,\, J-1) - h_1(i;\, m,\, M,\, J) \right| \quad \text{for} \quad i = \overline{1, N}, \tag{103}$$

$$\max_{[0;\,1]} \left| h_1(i;\, m,\, M,\, J) - h_1(i;\, m,\, M+1,\, J+1) \right| \leqslant \max_{[0;\,1]} \left| h_1(i;\, m,\, M-1,\, J-1) - h_1(i;\, m,\, M,\, J) \right| \quad \text{for} \quad i = \overline{1, N}, \tag{104}$$

and

$$\max_{[0;\,1]} \left| h_2(i;\, j,\, M,\, J) - h_2(i;\, j,\, M+1,\, J) \right| \leqslant \max_{[0;\,1]} \left| h_2(i;\, j,\, M-1,\, J) - h_2(i;\, j,\, M,\, J) \right| \quad \text{for} \quad i = \overline{1, N}, \tag{105}$$

$$\max_{[0;\,1]} \left| h_2(i;\, j,\, M,\, J) - h_2(i;\, j,\, M,\, J+1) \right| \leqslant \max_{[0;\,1]} \left| h_2(i;\, j,\, M,\, J-1) - h_2(i;\, j,\, M,\, J) \right| \quad \text{for} \quad i = \overline{1, N}, \tag{106}$$

$$\max_{[0;\,1]} \left| h_2(i;\, j,\, M,\, J) - h_2(i;\, j,\, M+1,\, J+1) \right| \leqslant \max_{[0;\,1]} \left| h_2(i;\, j,\, M-1,\, J-1) - h_2(i;\, j,\, M,\, J) \right| \quad \text{for} \quad i = \overline{1, N}, \tag{107}$$

along with

$$\left\| h_1(i;\, m,\, M,\, J) - h_1(i;\, m,\, M+1,\, J) \right\| \leqslant \left\| h_1(i;\, m,\, M-1,\, J) - h_1(i;\, m,\, M,\, J) \right\| \quad \text{in} \quad \mathbb{L}_2[0;\,1] \quad \text{for} \quad i = \overline{1, N}, \tag{108}$$

$$\left\| h_1(i;\, m,\, M,\, J) - h_1(i;\, m,\, M,\, J+1) \right\| \leqslant \left\| h_1(i;\, m,\, M,\, J-1) - h_1(i;\, m,\, M,\, J) \right\| \quad \text{in} \quad \mathbb{L}_2[0;\,1] \quad \text{for} \quad i = \overline{1, N}, \tag{109}$$

$$\left\| h_1(i; m, M, J) - h_1(i; m, M+1, J+1) \right\| \leqslant \left\| h_1(i; m, M-1, J-1) - h_1(i; m, M, J) \right\| \text{ in } \mathbb{L}_2[0; 1] \text{ for } i = \overline{1, N}, \quad (110)$$

and

$$\left\| h_2(i; j, M, J) - h_2(i; j, M+1, J) \right\| \leqslant \left\| h_2(i; j, M-1, J) - h_2(i; j, M, J) \right\| \text{ in } \mathbb{L}_2[0; 1] \text{ for } i = \overline{1, N}, \quad (111)$$

$$\left\| h_2(i; j, M, J) - h_2(i; j, M, J+1) \right\| \leqslant \left\| h_2(i; j, M, J-1) - h_2(i; j, M, J) \right\| \text{ in } \mathbb{L}_2[0; 1] \text{ for } i = \overline{1, N}, \quad (112)$$

$$\left\| h_2(i; j, M, J) - h_2(i; j, M+1, J+1) \right\| \leqslant \left\| h_2(i; j, M-1, J-1) - h_2(i; j, M, J) \right\| \text{ in } \mathbb{L}_2[0; 1] \text{ for } i = \overline{1, N}, \quad (113)$$

are required.

Definition 8. An approximate equilibrium (59) in staircase game (12) is called probability-$\{M, J\}$-consistent if inequalities (102)—(113) hold. The set of probabilities (75) of support (74) is called first-player-probability-$\{M, J\}$-consistent if inequalities (102)—(104) and (108)—(110) hold. The set of probabilities (77) of support (76) is called second-player-probability-$\{M, J\}$-consistent if inequalities (105)—(107) and (111)—(113) hold.

In accordance with **Definitions 3—8**, a player's equilibrium strategy (or its support, or the support probabilities) may be consistent while an equilibrium strategy of the other player is not consistent. This is done intentionally because it is not worth cancelling the player's equilibrium strategy consistency when for the other player the consistency conditions do not hold.

If inequalities (68)—(73), (78)—(83), (90)—(95), (102)—(113) hold for some i, then bimatrix game (49), assigned to the subinterval between $\tau^{(i-1)}$ and $\tau^{(i)}$, has a weakly consistent approximate solution to the corresponding continuous game (27) by (28)—(33). On this basis, the weak consistency of an approximate solution to a staircase game (12) is formulated.

Definition 9. The stack of successive equilibria (59) is called a weakly $\{M, J\}$-consistent approximate solution of staircase game (12) if inequalities (68)—(73), (78)—(83), (90)—(95), (102)—(113) hold. Stack (60) is called weakly first-player-$\{M, J\}$-consistent if inequalities (68), (70), (72), (78)—(80), (90)—(92), (102)—(104), (108)—(110) hold. Stack (61) is called weakly second-player-$\{M, J\}$-consistent if inequalities (69), (71), (73), (81)—(83), (93)—(95), (105)—(107), (111)—(113) hold.

Similarly to strengthening **Definitions 4** and **6**, the weak consistency can be strengthened by adding the requirements with inequalities (84)—(89) and (96)—(101).

Definition 10. The stack of successive equilibria (59) is called an $\{M, J\}$-consistent approximate solution of staircase game (12) if inequalities (68)—(73) and (78)—(113) hold. Stack (60) is called first-player-$\{M, J\}$-consistent if inequalities (68), (70), (72), (78)—(80), (84)—(86), (90)—(92), (96)—(98), (102)—(104), (108)—(110) hold. Stack (61) is called second-player-$\{M, J\}$-consistent if inequalities (69), (71), (73), (81)—(83), (87)—(89), (93)—(95), (99)—(101), (105)—(107), (111)—(113) hold.

As in the case of the zero-sum game [23], the approximate solution consistency theoretically proposes a better approximation than the weak consistency. The weak consistency notion by **Definition 9** may be thought of as it is decomposed by **Definitions 3**, **4**, **6**, **8**. Thus, the consistency notion by **Definition 10** is decomposed into **Definitions 3**, **5**, **7**, **8**.

Payoff consistency relaxation

Although there are six inequalities to be checked after solving seven bimatrix games on each subinterval, the payoff consistency is checked the easiest and fastest. Even if an approximate solution is not weakly consistent, it may be, e. g., payoff-consistent. A payoff-consistent solution can be sufficient to accept it as an appropriate approximate solution [1, 2, 14, 24]. However, if a one of $6N$ inequalities (68)—(73) is violated, even this type of consistency does not work. Meanwhile, the violation may be induced by a very small growth of the payoff change at a player (on a subinterval). Therefore, it is useful and practically reasonable to consider the payoff consistency adding a relaxation to inequalities (68)—(73).

Definition 11. An approximate equilibrium (59) in staircase game (12) is called ε-payoff-$\{M, J\}$-consistent if inequalities

$$\left| v_i^*(M, J) - v_i^*(M+1, J) \right| - \varepsilon \leqslant \left| v_i^*(M-1, J) - v_i^*(M, J) \right| \text{ by some } \varepsilon > 0 \text{ for } i = \overline{1, N}, \quad (114)$$

$$\left|z_i^*(M, J) - z_i^*(M+1, J)\right| - \varepsilon \leqslant \left|z_i^*(M-1, J) - z_i^*(M, J)\right|$$

$$\text{by some } \varepsilon > 0 \text{ for } i = \overline{1, N}, \tag{115}$$

$$\left|v_i^*(M, J) - v_i^*(M, J+1)\right| - \varepsilon \leqslant \left|v_i^*(M, J-1) - v_i^*(M, J)\right|$$

$$\text{by some } \varepsilon > 0 \text{ for } i = \overline{1, N}, \tag{116}$$

$$\left|z_i^*(M, J) - z_i^*(M, J+1)\right| - \varepsilon \leqslant \left|z_i^*(M, J-1) - z_i^*(M, J)\right|$$

$$\text{by some } \varepsilon > 0 \text{ for } i = \overline{1, N}, \tag{117}$$

$$\left|v_i^*(M, J) - v_i^*(M+1, J+1)\right| - \varepsilon \leqslant \left|v_i^*(M-1, J-1) - v_i^*(M, J)\right|$$

$$\text{by some } \varepsilon > 0 \text{ for } i = \overline{1, N}, \tag{118}$$

$$\left|z_i^*(M, J) - z_i^*(M+1, J+1)\right| - \varepsilon \leqslant \left|z_i^*(M-1, J-1) - z_i^*(M, J)\right|$$

$$\text{by some } \varepsilon > 0 \text{ for } i = \overline{1, N} \tag{119}$$

hold. Stack (60) is called first-player-ε-payoff-$\{M, J\}$-consistent if inequalities (114), (116), (118) hold. Stack (61) is called second-player-ε-payoff-$\{M, J\}$-consistent if inequalities (115), (117), (119) hold.

To ascertain whether the stack of successive equilibria (59) is weakly consistent or not, the seven bunches of N bimatrix games (49) should be solved, where the sampling density is defined by integers

$$\{M-1, J-1\},\ \{M-1, J\},\ \{M, J-1\},\ \{M, J\},$$
$$\{M+1, J\},\ \{M, J+1\},\ \{M+1, J+1\}.$$

It is worth noting once again that the players select their respective integers M and J independently and, moreover, the sampling by an integer S means that those $S-2$ points within an open interval can be chosen in any way, not necessarily to be uniformly distributed through the interval. Only the requirement of the proper sampling increment (by **Definition 1**) is followed. Nevertheless, the consistency meant by some sampling density integers $\{M, J\}$ does not guarantee that both the players will select such sampling density. Moreover, it is hard to find a continuous 2-person game, for which a consistent approximate equilibrium could be determined at appropriately small integers M and J. However, it is quite naturally to expect that, as they are increased (i. e., the sampling is made denser), the approximate equilibria (stacked equilibria) must converge to the respective equilibrium of staircase game (12). Here, it is quite important to use the phrase “respective equilibrium” because the initial staircase game (12) may have multiple staircase equilibria or a continuum of staircase equilibria (although adjective “staircase” gives a hint to the type of the game, rather than to the structure of its equilibria, a player’s strategy in a staircase equilibrium is equivalent to a staircase function if the strategy is a stack of subinterval pure strategies; even when the stack has mixed strategies on some subintervals, the eventual view of the stacked strategy is staircase-like). Therefore, the most appropriate (e. g., profitable for both players) staircase equilibrium should be selected. Besides, the approximate equilibria must become “more” consistent, which means that more inequalities of the bunch of inequalities (68)—(73) and (78)—(113) must hold.

An example of 2-person game approximation

To give an example of 2-person game approximation, consider a case in which $t \in [0.1\pi;\ 0.9\pi]$, the set of pure strategies of the first player is

$$X = \left\{x(t),\ t \in [0.1\pi;\ 0.9\pi] : 4 \leqslant x(t) \leqslant 7\right\} \subset \mathbb{L}_2[0.1\pi;\ 0.9\pi] \tag{120}$$

and the set of pure strategies of the second player is

$$Y = \left\{y(t),\ t \in [0.1\pi;\ 0.9\pi] : 1.5 \leqslant y(t) \leqslant 7.5\right\} \subset \mathbb{L}_2[0.1\pi;\ 0.9\pi], \tag{121}$$

where each of the players is allowed to change its pure strategy value at time points

$$\left\{\tau^{(i)}\right\}_{i=1}^{7} = \left\{0.1\pi + 0.1\pi i\right\}_{i=1}^{7}. \tag{122}$$

The players’ payoff functionals are

$$K(x(t), y(t)) = \int_{[0.1\pi;\ 0.9\pi]} 2\sin^2\left(0.5xt + \frac{\pi}{8}\right) \times \sin^3\left(0.2yt - \frac{7\pi}{13}\right) e^{-0.015xt} d\mu(t) \tag{123}$$

and

$$H\left(x(t),\, y(t)\right) = \int\limits_{[0.1\pi;\ 0.9\pi]} 2\sin^2\left(0.25xt - \frac{\pi}{10}\right) \times \sin^2\left(1.05yt + \frac{4\pi}{5}\right) e^{0.021yt} d\mu(t). \quad (124)$$

So, each of the players possesses 8-subinterval staircase function-strategies defined on interval $[0.1\pi;\ 0.9\pi]$. Hence, the 2-person staircase game is represented as a succession of 8 2-person games (27)

$$\left\langle \left\{[4;\ 7],\, [1.5;\ 7.5]\right\},\, \left\{K\left(\alpha_i,\, \beta_i\right),\, H\left(\alpha_i,\, \beta_i\right)\right\} \right\rangle \quad (125)$$

by

$$\alpha_i = x(t) \in [4;\ 7] \text{ and } \beta_i = y(t) \in [1.5;\ 7.5] \quad \forall\, t \in [0.1\pi i;\ 0.1\pi + 0.1\pi i) \text{ for } i = \overline{1,\, 7} \text{ and } \forall\, t \in [0.8\pi;\ 0.9\pi], \quad (126)$$

where the factual payoff of the first player in situation (29) is

$$K\left(\alpha_i,\, \beta_i\right) = \int\limits_{[0.1\pi;\ 0.1\pi + 0.1\pi i)} 2\sin^2\left(0.5\alpha_i t + \frac{\pi}{8}\right) \times \sin^3\left(0.2\beta_i t - \frac{7\pi}{13}\right) e^{-0.015\alpha_i t} d\mu(t) \quad \forall\, i = \overline{1,\, 7} \quad (127)$$

and

$$K\left(\alpha_8,\, \beta_8\right) = \int\limits_{[0.8\pi;\ 0.9\pi]} 2\sin^2\left(0.5\alpha_8 t + \frac{\pi}{8}\right) \times \sin^3\left(0.2\beta_8 t - \frac{7\pi}{13}\right) e^{-0.015\alpha_8 t} d\mu(t), \quad (128)$$

and the factual payoff of the second player in situation (29) is

$$H\left(\alpha_i,\, \beta_i\right) = \int\limits_{[0.1\pi;\ 0.1\pi + 0.1\pi i)} 2\sin^2\left(0.25\alpha_i t - \frac{\pi}{10}\right) \times \sin^2\left(1.05\beta_i t + \frac{4\pi}{5}\right) e^{0.021\beta_i t} d\mu(t) \quad \forall\, i = \overline{1,\, 7} \quad (129)$$

and

$$H\left(\alpha_8,\, \beta_8\right) = \int\limits_{[0.8\pi;\ 0.9\pi]} 2\sin^2\left(0.25\alpha_8 t - \frac{\pi}{10}\right) \sin^2\left(1.05\beta_8 t + \frac{4\pi}{5}\right) e^{0.021\beta_8 t} d\mu(t). \quad (130)$$

The first player's payoff functional (123) on each subinterval of set

$$\left\{ \left\{[0.1\pi i;\ 0.1\pi + 0.1\pi i)\right\}_{i=1}^{7},\, [0.8\pi;\ 0.9\pi] \right\} \quad (131)$$

is shown in **Fig. 2**. Compared to the second player's payoff functional (124) on each subinterval of set (131) shown in **Fig. 3**, the first player's payoff is a slow-changing functional. On the first subinterval $[0.1\pi;\ 0.2\pi)$ it is roughly a plane. Then, as time goes by, the first player's payoff starts slowly varying. The second player's payoff on the first subinterval is also a slow-varying function. As time goes by, it starts fluctuating – the closer the end is, the more waves it has.

The irregularity (non-uniformity) in the sampling is modelled as follows:

$$a_0^{(m)} = 4 + \frac{3m - 3}{M - 1} \text{ and } a^{(m)} = a_0^{(m)} + \frac{\xi_1}{M} \quad \text{for } m = \overline{2,\, M - 1} \quad (132)$$

by $a^{(1)} = 4$, $a^{(M)} = 7$, and

$$b_0^{(j)} = 1.5 + \frac{6j - 6}{J - 1} \text{ and } b^{(j)} = b_0^{(j)} + \frac{\xi_2}{J} \quad \text{for } j = \overline{2,\, J - 1} \quad (133)$$

by $b^{(1)} = 1.5$, $b^{(J)} = 7.5$, where ξ_1 and ξ_2 are values of two independent random variables distributed normally with zero mean and unit variance. The values resulting from (132) and (133) are sorted in ascending order, whereupon they are checked whether (40) and (42) are true. When either integer M or J is increased by 1, samplings (40) and (42) are checked whether they satisfy the proper sampling increment by **Definition 1**, i. e. whether inequality (48) holds for samplings (47) and (46).

Thus, 8 bimatrix games (49) with the players' payoff matrices (50) and (53) are formed from 8 2-person games (125), where

$$k_{imj}(M,\, J) = \int\limits_{[0.1\pi;\ 0.1\pi + 0.1\pi i)} 2\sin^2\left(0.5a^{(m)}t + \frac{\pi}{8}\right) \times \sin^3\left(0.2b^{(j)}t - \frac{7\pi}{13}\right) e^{-0.015a^{(m)}t} d\mu(t) \quad \text{for } i = \overline{1,\, 7}, \quad (134)$$

$$k_{8mj}(M, J) = \int_{[0.8\pi;\, 0.9\pi]} 2\sin^2\left(0.5a^{(m)}t + \frac{\pi}{8}\right) \times \sin^3\left(0.2b^{(j)}t - \frac{7\pi}{13}\right) e^{-0.015a^{(m)}t} d\mu(t) \qquad (135)$$

and

$$h_{imj}(M, J) = \int_{[0.1\pi;\, 0.1\pi+0.1\pi i)} 2\sin^2\left(0.25a^{(m)}t - \frac{\pi}{10}\right) \times \sin^2\left(1.05b^{(j)}t + \frac{4\pi}{5}\right) e^{0.021b^{(j)}t} d\mu(t) \quad \text{for} \quad i = \overline{1, 7}, \qquad (136)$$

$$h_{8mj}(M, J) = \int_{[0.8\pi;\, 0.9\pi]} 2\sin^2\left(0.25a^{(m)}t - \frac{\pi}{10}\right) \times \sin^2\left(1.05b^{(j)}t + \frac{4\pi}{5}\right) e^{0.021b^{(j)}t} d\mu(t). \qquad (137)$$

Although the subinterval length in (134)—(137) does not change, every subinterval has its "own" bimatrix game due to time variable t is explicitly included into the functions under the integral. This means that, as time goes by, the players develop their actions subinterval by subinterval.

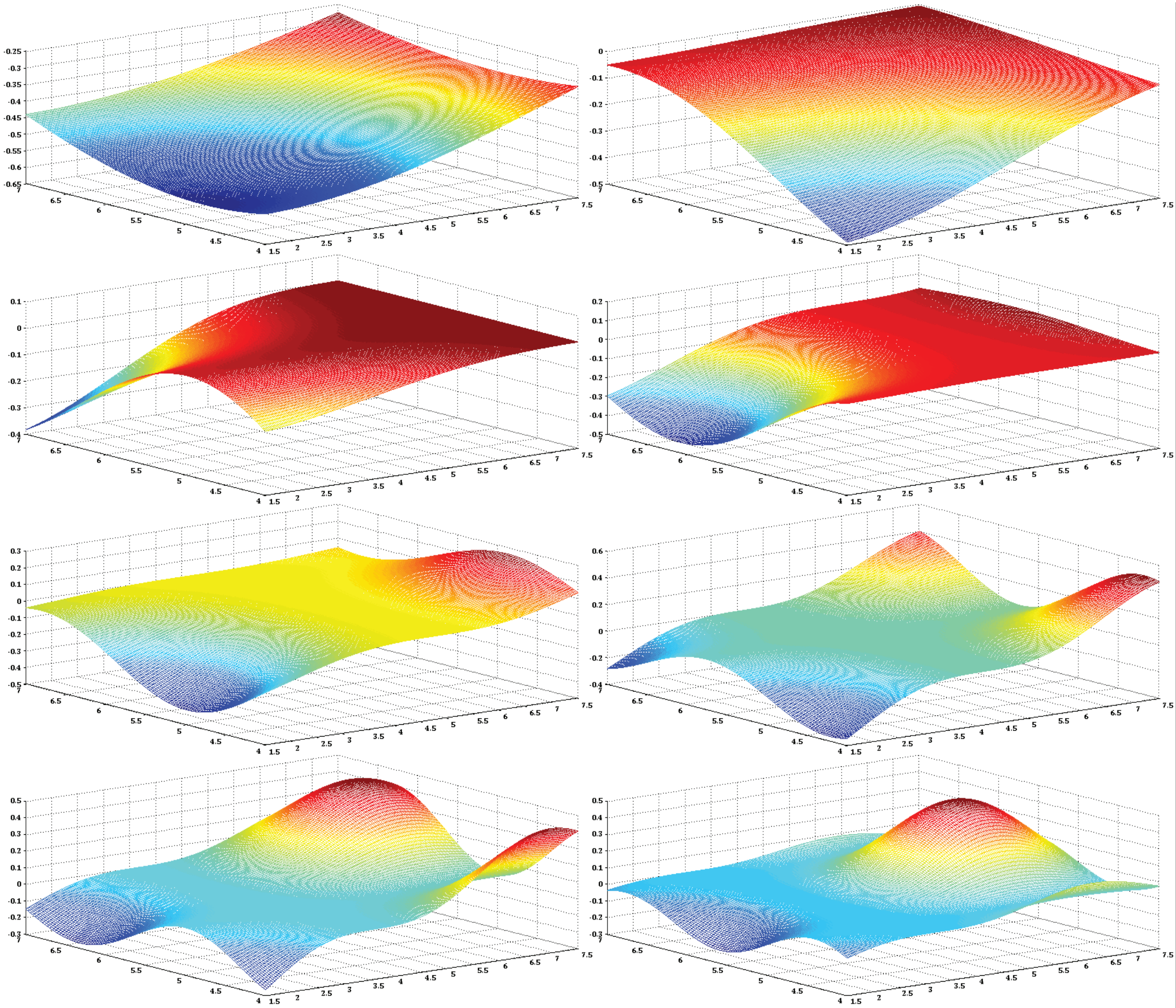

Fig. 2. The first player's payoff kernels (127), (128) on the 8 subintervals of set (131)

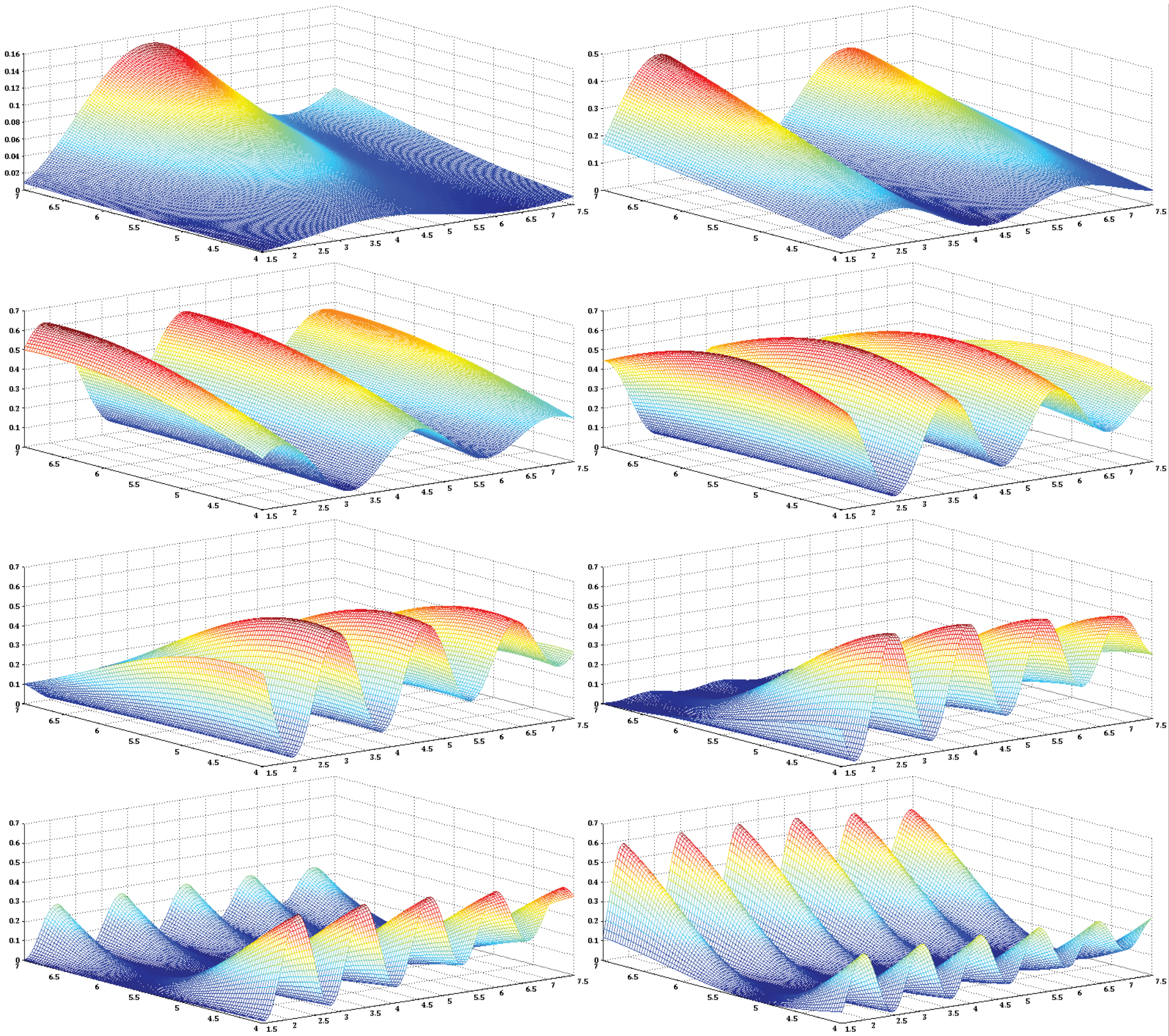

Fig. 3. The second player's payoff kernels (129), (130) on the 8 subintervals of set (131)

One of the trickiest problems with bimatrix games consists in multiple equilibria. To select a single equilibrium on each subinterval, a selection criterion should be defined. Thus, let an equilibrium be selected, by which the sum of the players' payoffs is maximal. Only this equilibrium will be visualized and discussed below.

Surely, the equilibrium solutions of these games (and the equilibrium solution of the initial staircase game) badly depend on the sampling. Subinterval-wise equilibrium strategies of the players by the sampling for every $M = \overline{3, 10}$ and $J = \overline{3, 10}$ are shown in **Fig. 4** in an indistinguishable bunch. In general, it is well seen that as the sampling density changes at such a relatively wide range of small sampling integers M and J, the player's equilibrium strategy (in every subinterval game, let alone the stacked optimal strategy on interval $[0.1\pi; 0.9\pi]$) badly varies. The only exception is the first, second, and fourth subintervals, on which the equilibrium strategies are pure and they do not change. Thus, the first player's equilibrium strategy on subintervals

$$[0.1\pi; 0.2\pi), \ [0.2\pi; 0.3\pi)$$

is

$$x^*(t) = 7 \quad \forall\, t \in [0.1\pi; 0.3\pi)$$

and it is

$$x^*(t) = 4 \quad \forall\, t \in [0.4\pi; 0.5\pi). \qquad (138)$$

The single stable subinterval equilibrium strategy of the second player is

$$y^*(t) = 1.5 \quad \forall t \in [0.4\pi; 0.5\pi). \tag{139}$$

The first player's payoff $v_i^*(M, J)$ (at the end of the i-th subinterval) and the payoff cumulative sum

$$v^{(n)*}(M, J) = \sum_{i=1}^{n} v_i^*(M, J) \text{ by } n = \overline{1, 8} \tag{140}$$

are scattered worse than the second player's payoff $z_i^*(M, J)$ and the payoff cumulative sum

$$z^{(n)*}(M, J) = \sum_{i=1}^{n} z_i^*(M, J) \text{ by } n = \overline{1, 8} \tag{141}$$

(**Fig. 5**), where

$$v^*(M, J) = v^{(8)*}(M, J) \tag{142}$$

and

$$z^*(M, J) = z^{(8)*}(M, J) \tag{143}$$

are the players' equilibrium payoffs in this staircase game.

It is noteworthy that during the first four subintervals there is a single pure strategy equilibrium in the subinterval bimatrix game, whichever the sampling is (so the above-mentioned criterion of the payoff sum maximization is not applied here at all). This fact is seen in **Fig. 5** also as the payoffs are less scattered by $t \in [0.1\pi; 0.5\pi]$. So, all the equilibria on half-interval $[0.1\pi; 0.5\pi)$ are in pure strategies, and only during the second half the pure-strategy "mixing" works. There appear multiple equilibria during that half, and the payoff sum maximization criterion is applied to select the best equilibrium point on the subinterval (at given M and J).

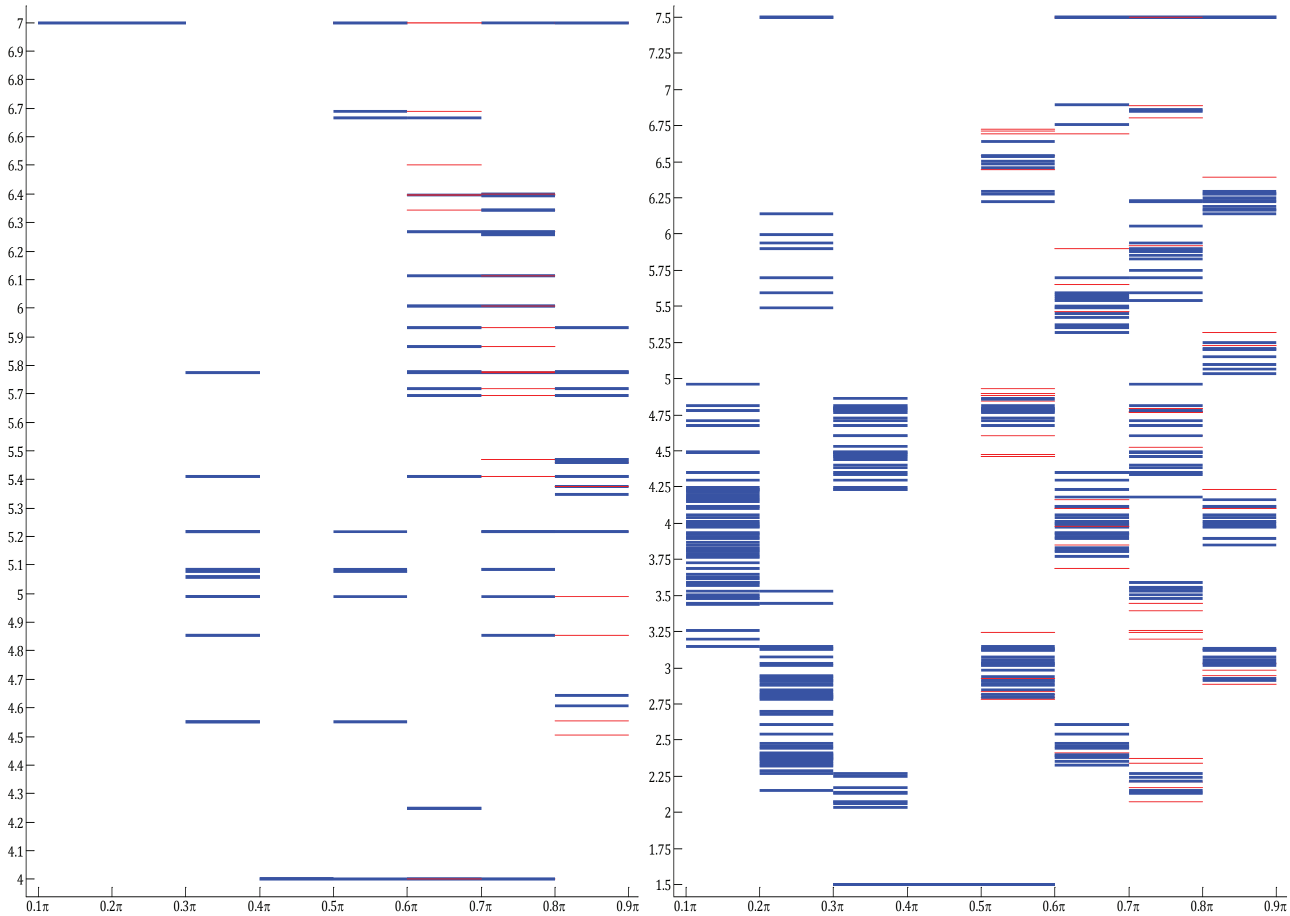

Fig. 4. An indistinguishable bunch of subinterval-wise optimal strategies of the first (left) and second (right) players by $M = \overline{3, 10}$ and $J = \overline{3, 10}$ (here and further below the equilibrium pure strategy is represented by thicker line, pure strategies from the mixed equilibrium strategy support are represented by thinner lines)

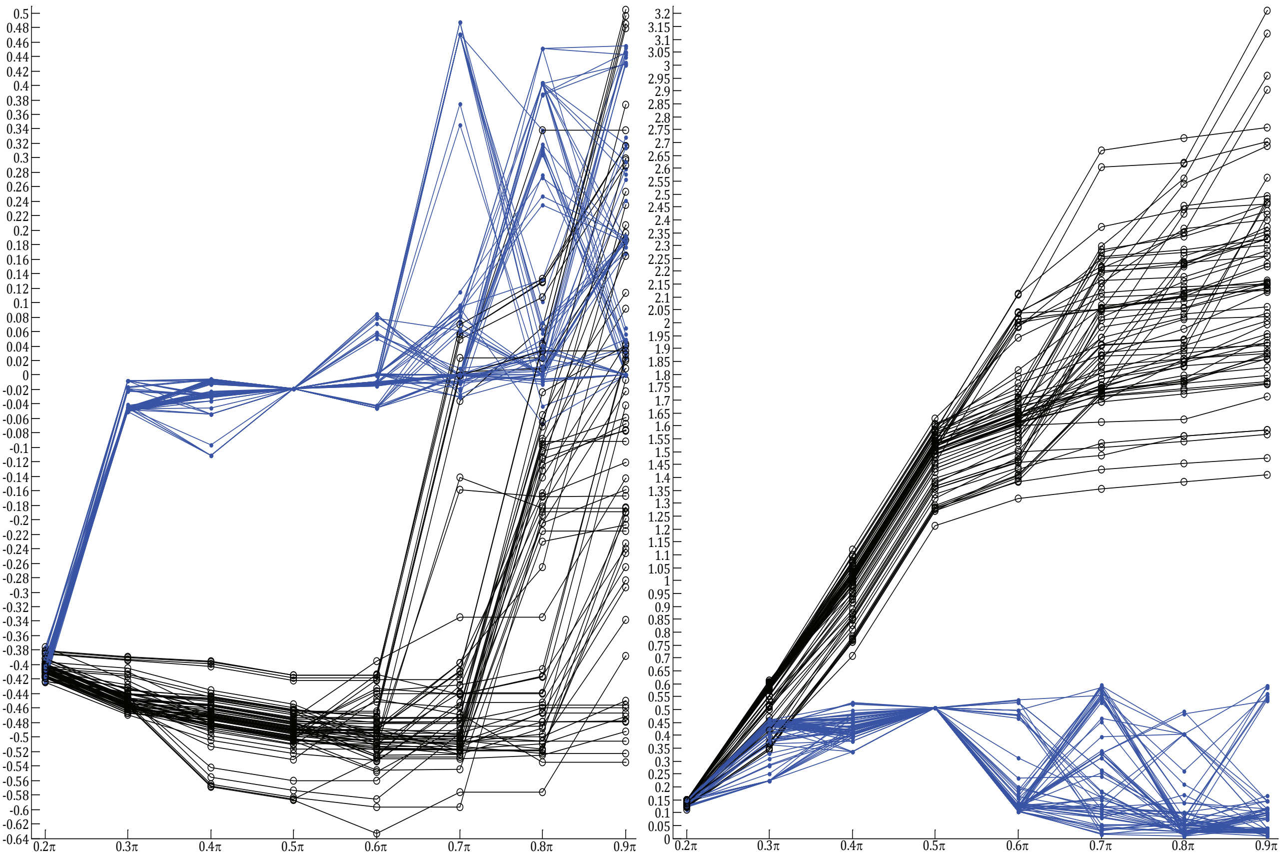

Fig. 5. An indistinguishable bunch of the first player's payoffs (left) and second player's payoffs (right) at the end of every subinterval (dots) and their cumulative sum (circles) by $M = \overline{3, 10}$ and $J = \overline{3, 10}$

As the sampling density is further increased up to solving 20×20 bimatrix games, subinterval equilibrium strategies (both pure and mixed) become more "condensed" (**Fig. 6**), as well as the subinterval payoffs and payoffs (140)–(143) do (**Fig. 7**). During the first four subintervals (the first half-interval from $t = 0.1\pi$ to $t = 0.5\pi$) there still is a single pure strategy equilibrium in the subinterval bimatrix game, whichever the sampling is. The first player's equilibrium strategies on the first and fourth subintervals are immobile: they are still

$$x^*(t) = 7 \quad \forall\, t \in [0.1\pi;\, 0.2\pi) \qquad (144)$$

and (138). The single stable subinterval equilibrium strategy of the second player is (139). So, there is the immobile pure strategy equilibrium point on the fourth subinterval consisting of (138) and (139). It is remarkable that the payoff cumulative sums at the end of the fourth subinterval are like to make a bundle (compare **Fig. 7** to **Fig. 5** at $t = 0.5\pi$).

Nevertheless, the first player's equilibrium payoffs in this staircase game appear to be badly scattered in a really wide range. It is likely that the growing multiplicity of equilibria influences (for instance, there are 187 equilibria on the last subinterval over all 64 versions of the sampling, whereas there are just 85 equilibria by $M = \overline{3, 10}$ and $J = \overline{3, 10}$). Although the range in **Fig. 7** is narrower than that in **Fig. 5** (see the vertical line of circles at $t = 0.9\pi$), the result is not satisfactory. This implies that the first player will definitely try to sample denser. The second player seems to do that too because the range of payoffs (143) is pretty wide also. So, as the sampling density is further increased up to solving 30×30 bimatrix games, the condensation of subinterval equilibrium strategies (**Fig. 8**) and payoffs (**Fig. 9**) progresses. The first player's equilibrium strategies on the first and fourth subintervals are still (144) and (138), whereas the single stable subinterval equilibrium strategy of the second player is (139). To state it in advance, this stable part of the staircase game does not change at all by any sampling.

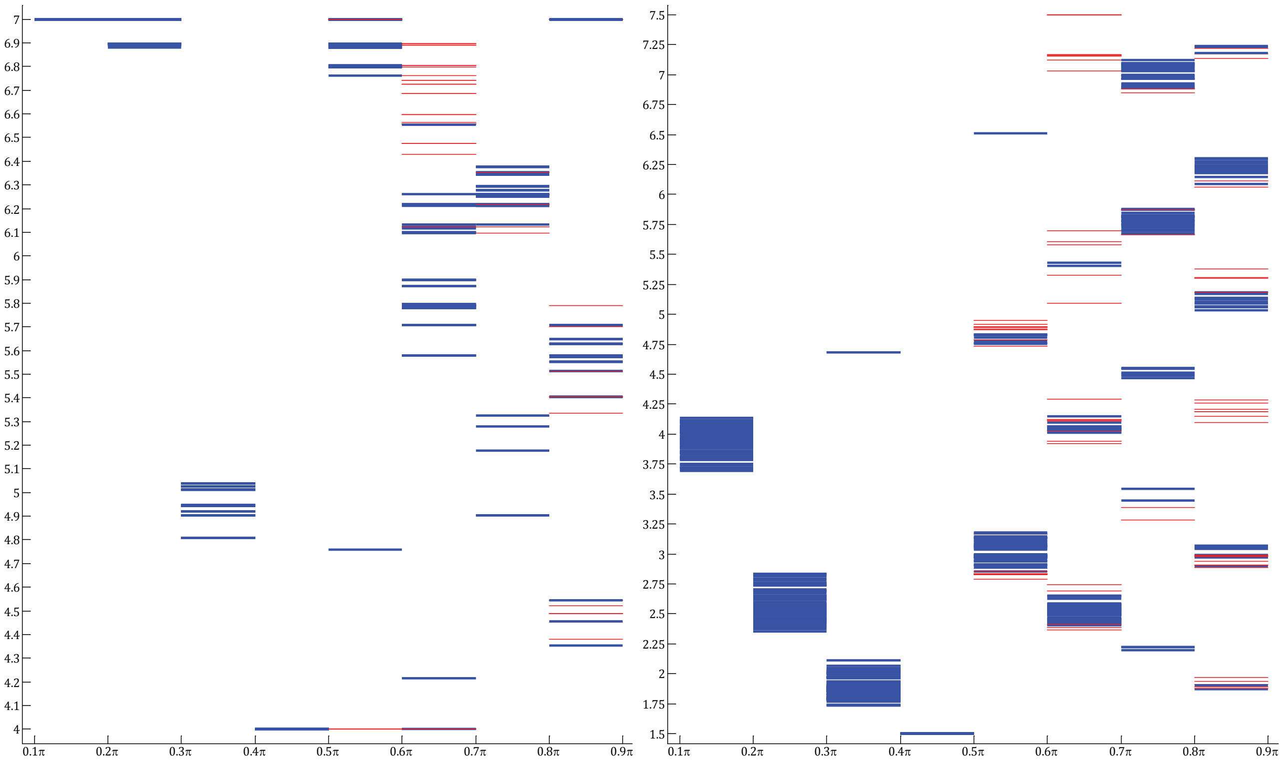

Fig. 6. A bunch of subinterval-wise optimal strategies of the first (left) and second (right) players by $M = \overline{13, 20}$ and $J = \overline{13, 20}$

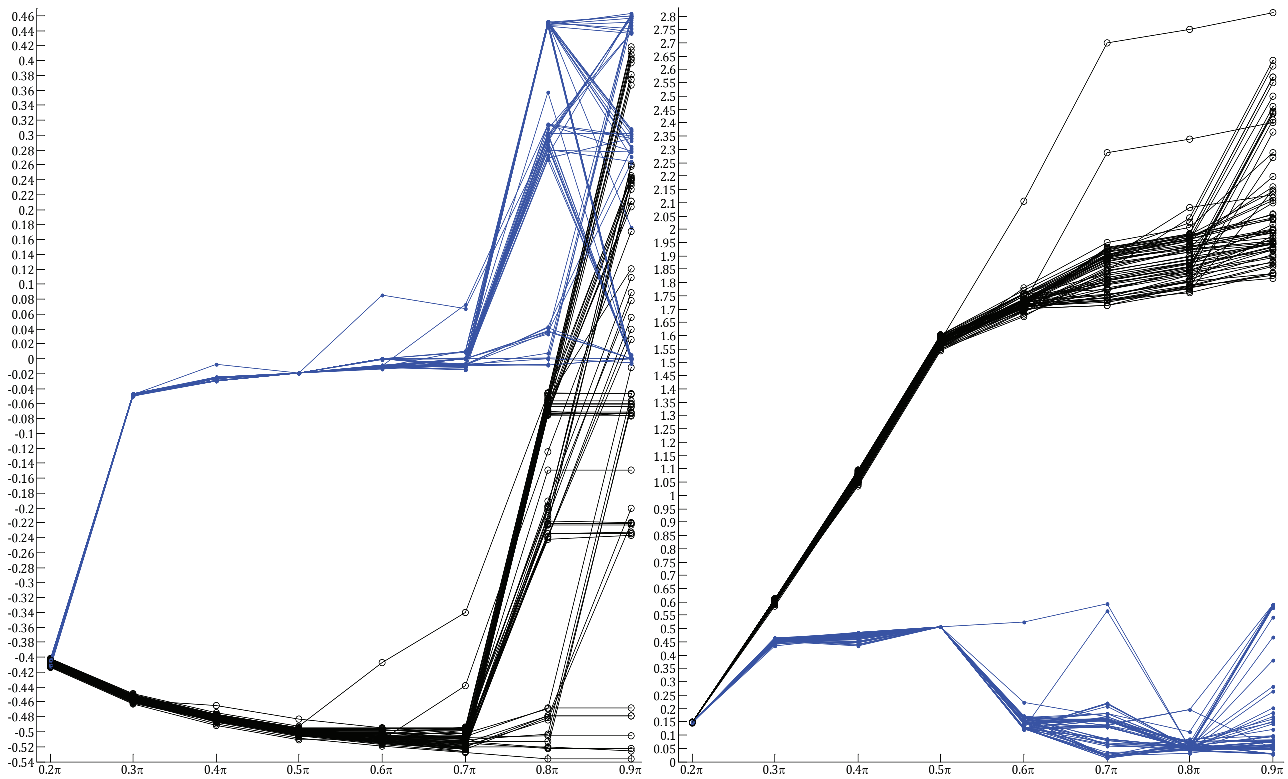

Fig. 7. A bunch of the first player's payoffs (left) and second player's payoffs (right) at the end of every subinterval (dots) and their cumulative sum (circles) by $M = \overline{13, 20}$ and $J = \overline{13, 20}$

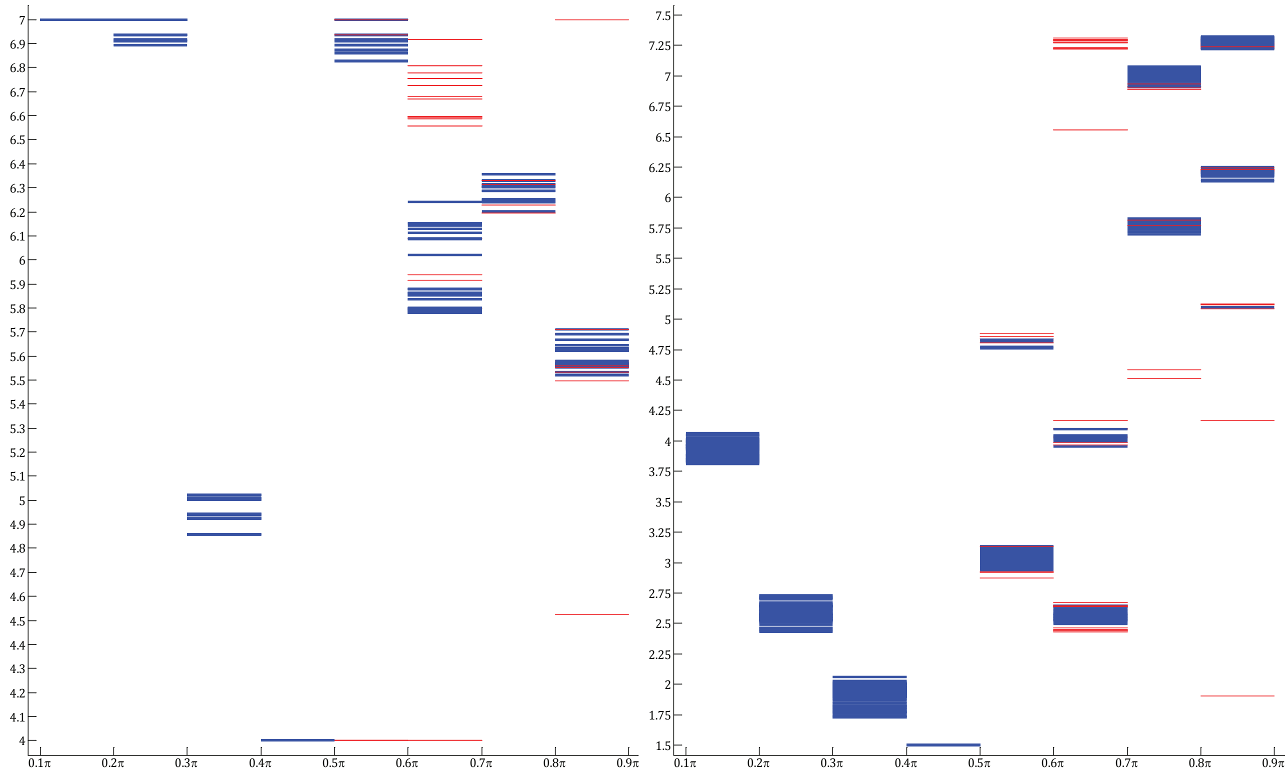

Fig. 8. A bunch of subinterval-wise optimal strategies of the first (left) and second (right) players by $M = \overline{23, 30}$ and $J = \overline{23, 30}$

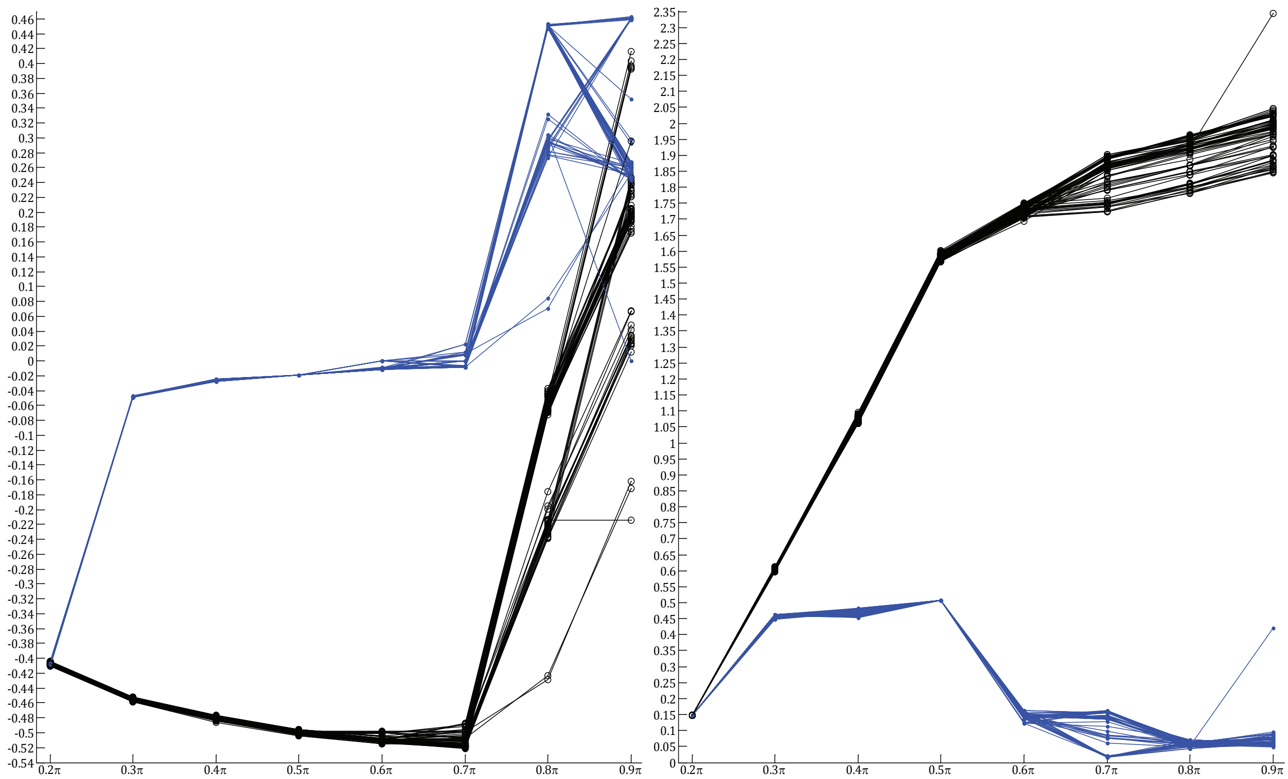

Fig. 9. A bunch of the first player's payoffs (left) and second player's payoffs (right) at the end of every subinterval (dots) and their cumulative sum (circles) by $M = \overline{23, 30}$ and $J = \overline{23, 30}$

Despite the growing multiplicity of equilibria (there are 192 equilibria on the last subinterval over all 64 versions of the sampling), now it is quite clear that the approximate equilibria (stacked equilibria) converge to the respective equilibrium of the staircase game by (120)–(124). This is easily seen by comparing **Fig. 4**, **6**, **8**. The convergence of payoffs is even clearer (**Fig. 5**, **7**, **9**). The second player's result seems almost satisfactory unlike that of the first player. It is noteworthy that the bunch of the first player's payoffs makes a tight bundle at the end of the sixth subinterval. This bundle is tighter than the bundle of the second player's payoffs at $t = 0.7\pi$ (**Fig. 9**).

Unfortunately, the players' equilibrium strategies (stacked equilibria) at these samplings are not even ε-payoff-{*M*, *J*}-consistent by sufficiently great ε. It is some paradoxical that the first player receives ε-consistent payoffs earlier than the second player does. However, this happens at an inappropriately big payoff consistency relaxation. The "paradox" is easily explained with that the range of the first player's payoff is far narrower with respect to that of the second player.

Will it be improved when the sampling is denser? Solving bigger games up to 40×40 bimatrix ones confirms the stacked equilibria convergence (**Fig. 10**). Compared to **Fig. 8**, no considerable changes in **Fig. 10** are visible. The same concerns the payoffs (**Fig. 11**), where the tight bundle of the first player's payoffs is seen at $t = 0.7\pi$. The problem with the payoff consistency remains, though. The first player's subinterval equilibrium strategies are ε-payoff-consistent on the first four subintervals by

$$\varepsilon = 0.0297 \cdot \left| v_i^*(M, J) \right| \quad \text{at} \quad i = \overline{1, 4}$$

by every

$$M = \overline{34, 39} \quad \text{and} \quad J = \overline{34, 39}.$$

So, if the staircase game was defined on just interval $[0.1\pi; 0.5\pi]$, the first player's stacked equilibrium strategies would be ε-payoff-{*M*, *J*}-consistent.

Further increasing sampling density (thickening the samplings) does not make sense: stacked equilibria do not change (compare **Fig. 12** to **Fig. 10**) and the payoffs remain with almost the same ranges (compare **Fig. 13** to **Fig. 11**). If the staircase game was defined on just interval $[0.1\pi; 0.5\pi]$, the first player's stacked equilibrium strategies would be ε-payoff-{*M*, *J*}-consistent by

$$\varepsilon = 0.016 \cdot \left| v_i^*(M, J) \right| \quad \text{at} \quad i = \overline{1, 4}$$

by every

$$M = \overline{44, 49} \quad \text{and} \quad J = \overline{44, 49}.$$

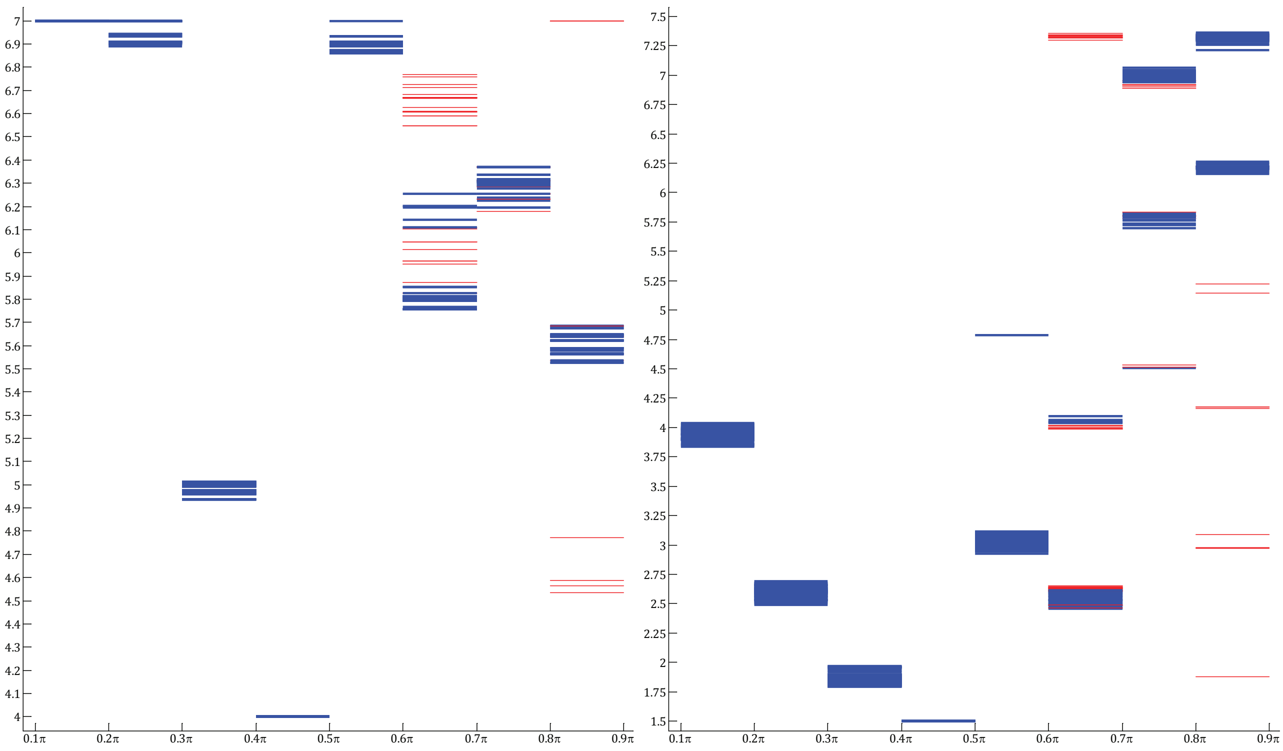

Fig. 10. A bunch of subinterval-wise optimal strategies of the first (left) and second (right) players by $M = \overline{33, 40}$ and $J = \overline{33, 40}$

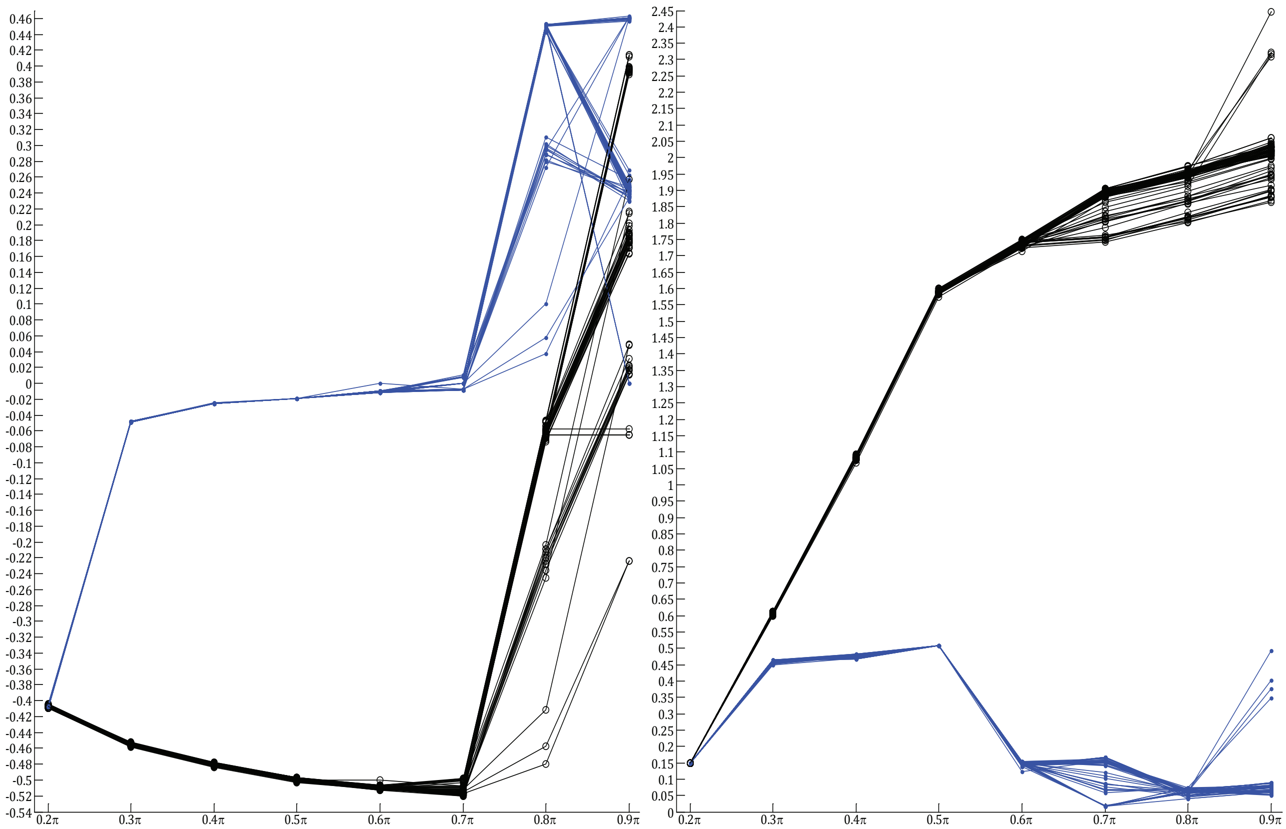

Fig. 11. A bunch of the first player's payoffs (left) and second player's payoffs (right) at the end of every subinterval (dots) and their cumulative sum (circles) by $M=\overline{33, 40}$ and $J=\overline{33, 40}$

If it was interval $[0.1\pi;\ 0.6\pi]$, the first player's stacked equilibrium strategies would be ε-payoff-{*M*, *J*}-consistent by

$$\varepsilon = 0.2016 \cdot \left|v_i^*(M, J)\right| \quad \text{at} \quad i=\overline{1, 5}$$

within the same samplings. Herein, if

$$\varepsilon = 0.1073 \cdot \left|z_i^*(M, J)\right| \quad \text{at} \quad i=\overline{1, 5},$$

the approximate equilibrium in the staircase game by (120)–(124) would be ε-payoff-{*M*, *J*}-consistent.

Eventually, this example shows that it may be very hard to find such an ε for which an approximate equilibrium in the staircase game would be ε-payoff-{*M*, *J*}-consistent. The matter is the range of payoffs of a player's may significantly differ from the range of the other player's payoffs. For instance, in the staircase game by (120)–(124), the first player's payoff varies roughly between –0.5772 and 0.4894, whereas the second player's payoff varies roughly between 0.0002 and 0.6408. Unlike the first player's payoff, the second player's payoff is always positive.

However, all the approximate equilibria in **Fig. 12** are ε-payoff-{*M*, *J*}-consistent by ε = 0.4888 (although it is too big payoff consistency relaxation). Moreover, every approximate equilibrium obtained by

$$M = J \in \{44, 45, 46, 47, 48, 49\}$$

is ε-payoff-{*M*, *M*}-consistent by ε = 0.3553, whereas every approximate equilibrium obtained by

$$M = J \in \{46, 47, 48, 49\}$$

is ε-payoff-{*M*, *M*}-consistent by ε = 0.2043, which is relatively not that bad. Although the solution convergence is apparent, the players' equilibrium strategies will not produce more consistent payoffs by further thickening the samplings. This is an evidence of that the solution convergence has reached its saturation, and further thickening the samplings will not improve the solution approximation nor improve the consistency. Therefore, the approximate solution to the 2-person staircase game by (120)–(124) can be accepted by the independent sampling at both players with the integers between 23 and 40 (of course, not necessarily identical).

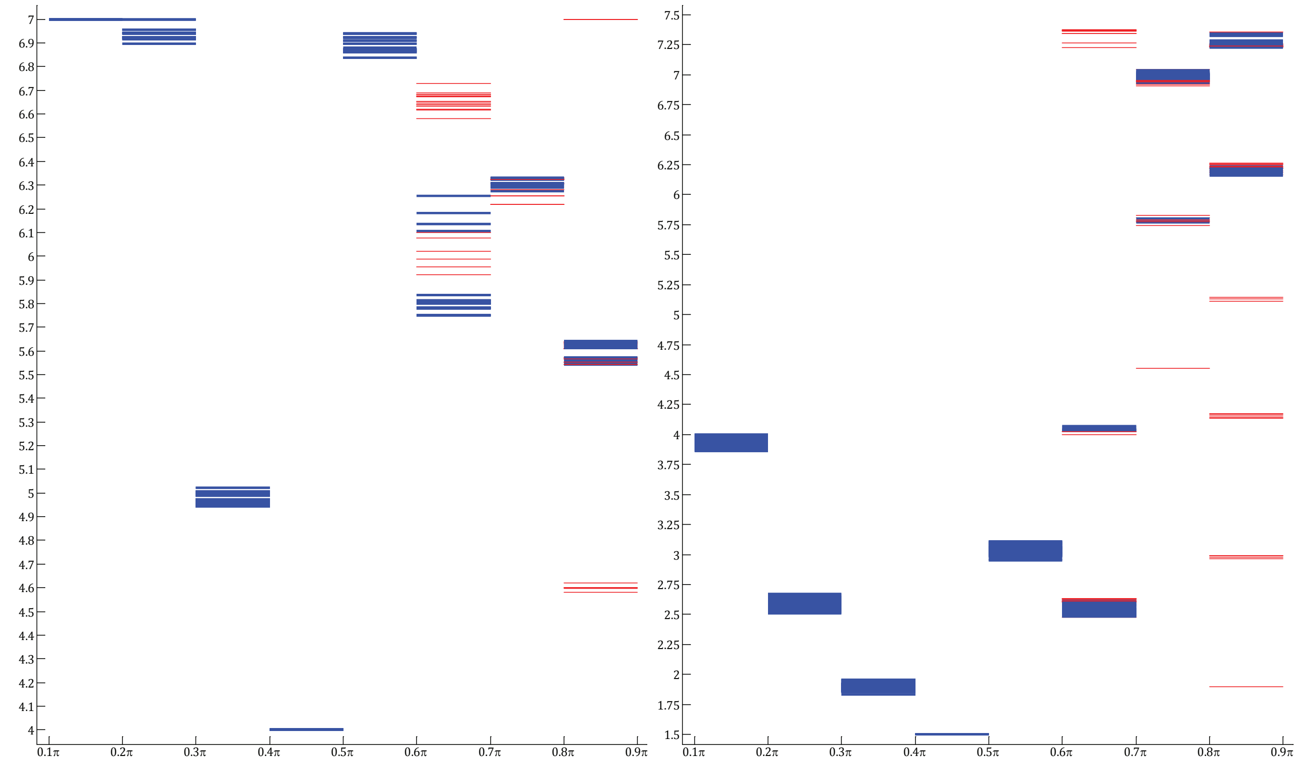

Fig. 12. A bunch of subinterval-wise optimal strategies of the first (left) and second (right) players by $M = \overline{43, 50}$ and $J = \overline{43, 50}$

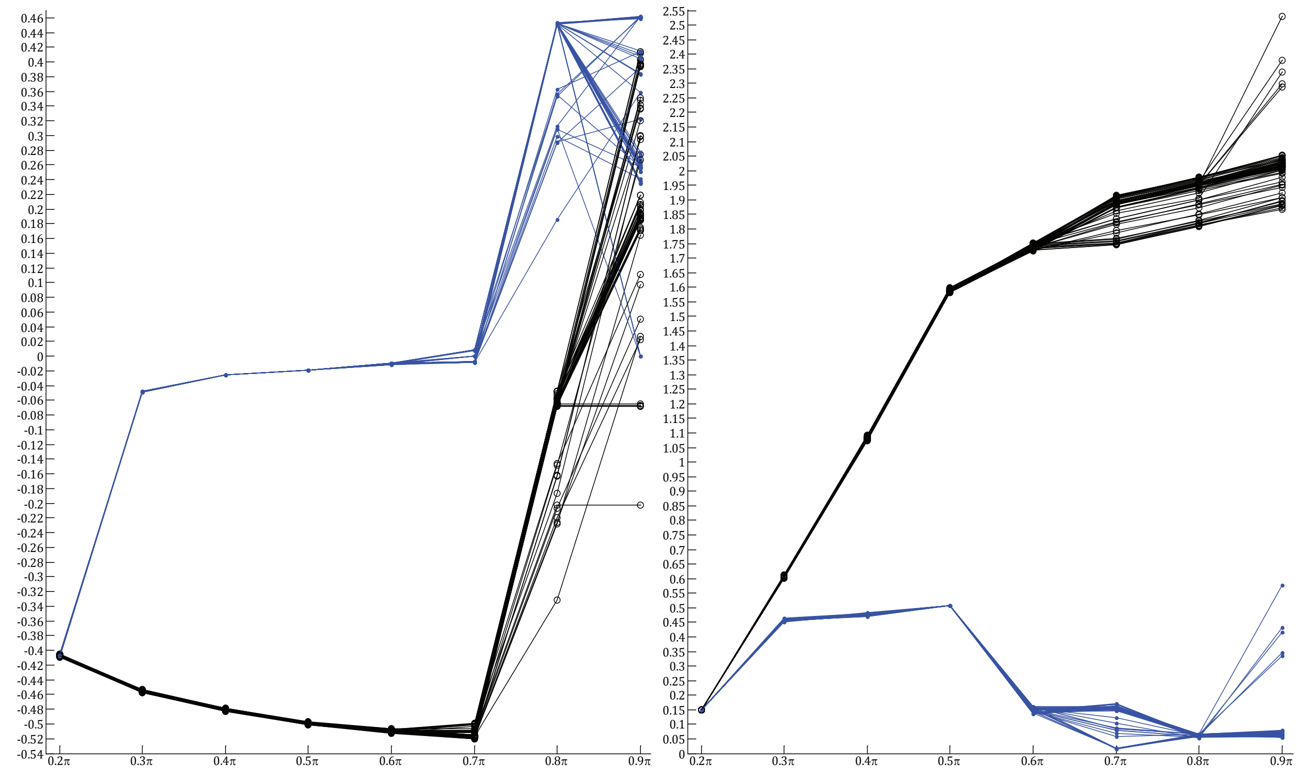

Fig. 13. A bunch of the first player's payoffs (left) and second player's payoffs (right) at the end of every subinterval (dots) and their cumulative sum (circles) by $M = \overline{43, 50}$ and $J = \overline{43, 50}$

Discussion of the contribution

Without considering each subinterval bimatrix game separately, it would be intractable to straightforwardly solve the sampled staircase game. For instance, by sampling the 2-person staircase game given by (120)—(124), where each of the players uses 8-subinterval staircase function-strategies, even with, say, $M = 24$ and $J = 23$, the resulting $24^8 \times 23^8$ bimatrix game (in which, e. g., the first player has 110075314176 pure strategies) cannot be solved in a reasonable time span. Indeed, it is at least hard to store those more than 110 billion pure strategies, let alone processing them (in searching for equilibria). Therefore, solving subinterval bimatrix games (which are obviously "smaller") separately and then stacking (or stitching, in more understandable terms) their solutions is a far more efficient way to obtain an approximate solution of the initial staircase game. The applicability of this method may be limited to the subinterval bimatrix game size defined by M and J. For instance, the computation time has an exponentially-increasing dependence on the size of the square matrix. Solving bimatrix games, in which each of the players has at least a few hundred pure strategies, may be time-consuming in applications requiring fast updates of the solution (when the structure of the initial staircase game changes itself).

The (weak) consistency of an approximate solution is a criterion of its acceptability. However, a (weakly) consistent approximate solution may not exist at appropriately small (tractable) M and J. So, the consistency decomposition into parts by **Definitions 3—8** and particularly isolating an ε-payoff consistency by **Definition 11** is justified and practically applicable.

There are still many open questions, though. First, the requirement of the proper sampling increment (**Definition 1**) given by strict inequality (48) may seem not enough rigorous. The matter is that it cannot guarantee that the sampled points in a 1-incremented sampling will be closer to each other (see **Fig. 1**). However, the respective requirement in the form of inequality

$$\max_{s=\overline{1, S}} \left(\lambda^{(s+1)} - \lambda^{(s)} \right) < \min_{s=\overline{1, S-1}} \left(\zeta^{(s+1)} - \zeta^{(s)} \right) \quad (145)$$

guaranteeing the mentioned property appears to be too rigorous. Indeed, if the player follows (145), the proper sampling increment is going to fail if there is a pair of too close points in the previous sampling.

Second, it is not proved that limits

$$\lim_{M \to \infty,\, J \to \infty} v_i^* (M, J) \quad \forall\, i = \overline{1, N} \quad (146)$$

and

$$\lim_{M \to \infty,\, J \to \infty} z_i^* (M, J) \quad \forall\, i = \overline{1, N} \quad (147)$$

exist and they are equal to the respective equilibrium values of the subinterval continuous games. Third, if limits (146) and (147) exist, it is not proved that this is followed by that any approximate equilibrium (59) is ε-payoff-$\{M, J\}$-consistent for any $M \geqslant M_*$ and $J \geqslant J_*$ ($M_* \in \mathbb{N} \setminus \{1\}$, $J_* \in \mathbb{N} \setminus \{1\}$), let alone the problem of the equilibria multiplicity. The inter-influence among the consistency decomposition parts by **Definitions 3—8** is also uncertain yet.

The question of a possible reconciliation of the difference of the players' sampling step selection is indeed that hard. The players can select their samplings simultaneously but identical samplings are of small likelihood. Even if the ranges of function-strategy values are identical and sampling integers M and J are the same (i. e., $M = J$), implying the uniform samplings, a player's sampling may differ from the other player's sampling due to eventual inaccuracies in selecting points. In the example of 2-person game approximation, this has been modelled by (132) and (133) with using normal "noise" in the point selection. However, at sufficiently great sampling integers M and J, not necessarily equal, significant changes in M and J are expected not to influence the approximate solution much (see **Fig. 8**, **10**, **12**, and **Fig. 9**, **11**, **13**). Just like in the above-considered example, the player's equilibrium strategies converge subinterval-wise and the resulting staircase strategy appears to be an acceptable approximate equilibrium strategy in the initial staircase game.

Therefore, the presented method is a significant contribution to the 2-person game theory and its finite approximation supplement. It allows approximately solving 2-person games with staircase-function strategies in a far simpler manner regardless of the fact that the players may sample their sets of function-strategy values differently [14, 20, 24]. Once the (weak) consistency is confirmed (the respective approximate solution should be at least ε-payoff consistent by **Definition 11**), the approximate pure-mixed-strategy solution (like those ones of staircase strategies in **Fig. 8**, **10**, **12**) can be easily implemented and practiced [6, 7, 9, 12, 20].

Conclusion

A non-cooperative 2-person game played in staircase-function continuous spaces is approximated to a bimatrix game by sampling the players' pure strategy value sets. Each set is irregularly sampled in its own way so that the resulting samplings may be of different cardinalities and varying densities. While sampled, the requirement of the proper sampling increment (by **Definition 1**) must be followed – the $S+1$ points in a 1-incremented sampling must be selected denser than S points.

Owing to **Theorem 2**, the solution of the bimatrix game is obtained by stacking the solutions of the "smaller" ("shorter") bimatrix games, each defined on a subinterval where the pure strategy value is constant. In this research, the Nash equilibrium has been taken as the solution type, although some other types might be considered as well. However, this is a matter of future research.

The stack of the "smaller" bimatrix game equilibria is an approximate solution to the initial staircase game. The (weak) consistency of the approximate solution is studied by how much the payoff and equilibrium change as the sampling density minimally increases by the three ways of the sampling increment: only the first player's increment, only the second player's increment, both the players' increment. Thus, the consistency, equivalent to the approximate solution acceptability, is decomposed into the payoff (**Definition 3**), equilibrium strategy support cardinality (**Definitions 4** and **5**), equilibrium strategy sampling density (**Definitions 6** and **7**), and support probability consistency (**Definition 8**). The weak consistency itself is a relaxation to the consistency, where the minimal decrement of the sampling density is ignored. The suggested method of finite approximation of staircase 2-person games consists in the independent samplings, solving "smaller" bimatrix games, and stacking their solutions if they are consistent.

The most important part is the payoff consistency. It is checked in the quickest and easiest way. In practice, it is reasonable to consider a relaxed payoff consistency. The relaxed payoff consistency by (114)–(119) means that, as the sampling density minimally increases (in each of the three ways of the sampling increment), the game value change in an appropriate approximation may grow at most by ε for each of the players. The equilibrium strategy support cardinality (weak) consistency is checked even easier, but it takes some resource to calculate the support cardinality, whereas the payoff is received "instantly". In general, the relaxed payoff consistency is the main (and often the single) item to be controlled for the successful approximation. The finite approximation is regarded appropriate if at least the respective approximate (stacked) equilibrium is ε-payoff consistent (**Definition 11**).

One can notice that, in staircase game (12) decomposed into games (27), the player's payoff value depends only on the subinterval length if time t is not explicitly included into the function under the respective integral in (8) or (9). If the subinterval length does not change, and time t is not explicitly included into the functions (10), (11) under the integrals in (8), (9), then every subinterval has the same bimatrix game. The triviality of the equal-length-subinterval solution is explained by a standstill of the players' strategies. For instance, time variable t explicitly included into functional (8) means that the first player may develop one's actions due to the game-modelled system changes (develops) as time goes by.

Finite approximation of games played in staircase-function continuous spaces will be extended and advanced also for the case of three players sampling their strategy value sets irregularly. The independence of the player's sampling step selection may have a deeper incompatibility impact in the trimatrix game case, where the problem of the equilibria multiplicity and the varying profitability have far trickier consequences.

References

[1] N.N. Vorob'yov, *Game theory fundamentals. Noncooperative games*, Moscow, Nauka, 1984, 496 p. (in Russian).

[2] N.N. Vorob'yov, *Game theory for economists-cyberneticists*, Moscow, Nauka, 1985, 272 p. (in Russian).

[3] M.J. Osborne, *An introduction to game theory*, Oxford University Press, 2003, 554 p.

[4] N. Nisan *et al.*, *Algorithmic Game Theory*, Cambridge University Press, 2007, 778 p. Available: https://doi.org/10.1017/CBO9780511800481

[5] K. Leyton-Brown and Y. Shoham, *Essentials of game theory: a concise, multidisciplinary introduction*, Morgan & Claypool Publishers, 2008, 104 p. Available: https://doi.org/10.2200/S00108ED1V01Y200802AIM003

[6] R.B. Myerson, *Game theory: Analysis of Conflict*, Harvard University Press, 1997, 600 p.

[7] V.V. Romanuke, *Theory of Antagonistic Games*, Lviv, New World, 2000, 2010, 294 p. (in Ukrainian).

[8] V.V. Romanuke, "Finite approximation of continuous noncooperative two-person games on a product of linear strategy functional spaces", in *Journal of Mathematics and Applications*, 2020, vol. 43, pp. 123–138. Available: https://doi.org/10.7862/rf.2020.9

[9] T.C. Schelling, *The Strategy of Conflict*, Harvard University, 1980, 328 p.

[10] S. Rahal *et al.*, "Hybrid strategies using linear and piecewise-linear decision rules for multistage adaptive linear optimization", in *European Journal of Operational Research*, 2021, vol. 290, iss. 3, pp. 1014–1030. Available: https://doi.org/10.1016/j.ejor.2020.08.054

[11] S.P. Coraluppi and S. I. Marcus, "Risk-sensitive and minimax control of discrete-time, finite-state Markov decision processes", in *Automatica*, 1999, vol. 35 (2), pp. 301–309. Available: https://doi.org/10.1016/S0005-1098(98)00153-8

[12] H. Khaloie *et al.*, "Coordinated wind-thermal-energy storage offering strategy in energy and spinning reserve markets using a multi-stage model", in *Applied Energy*, 2020, vol. 259, 114168. Available: https://doi.org/10.1016/j.apenergy.2019.114168

[13] S. Kim *et al.*, "Flexible risk control strategy based on multi-stage corrective action with energy storage system", in *International Journal of Electrical Power & Energy Systems*, 2019, vol. 110, pp. 679–695. Available: https://doi.org/10.1016/j.ijepes.2019.03.064

[14] V.V. Romanuke, "Approximation of unit-hypercubic infinite two-sided noncooperative game via dimension-dependent irregular samplings and reshaping the multidimensional payoff matrices into flat matrices for solving the corresponding bimatrix game", in *Computer Modelling and New Technologies*, 2015, vol. 19, no. 3A, pp. 7–16.

[15] E.B. Yanovskaya, "Minimax theorems for games on the unit square", in *Probability theory and its applications*, 1964, no. 9 (3), pp. 554–555.

[16] S. Adlakha *et al.*, "Equilibria of dynamic games with many players: Existence, approximation, and market structure", in *Journal of Economic Theory*, 2015, vol. 156, pp. 269–316. Available: https://doi.org/10.1016/j.jet.2013.07.002

[17] P. Bernhard and J. Shinar, "On finite approximation of a game solution with mixed strategies", in *Applied Mathematics Letters*, 1990, vol. 3 (1), pp. 1–4. Available: https://doi.org/10.1016/0893-9659(90)90054-F

[18] E.B. Yanovskaya, "Antagonistic games played in function spaces", in *Lithuanian Mathematical Bulletin*, 1967, no. 3, pp. 547–557.

[19] J. Yang *et al.*, "Group formation in the spatial public goods game with continuous strategies", in *Physica A: Statistical Mechanics and its Applications*, 2018, vol. 505, pp. 737–743. Available: https://doi.org/10.1016/j.physa.2018.03.057

[20] V.V. Romanuke and V. G. Kamburg, "Approximation of isomorphic infinite two-person noncooperative games via variously sampling the players' payoff functions and reshaping payoff matrices into bimatrix game", in *Applied Computer Systems*, 2016, vol. 20, pp. 5–14. Available: https://doi.org/10.1515/acss-2016-0009

[21] R.E. Edwards, *Functional Analysis: Theory and Applications*, New York, Holt, Rinehart and Winston, 1965, 781 p.

[22] F.L. Lewis *et al.*, *Optimal Control*, Hoboken, New Jersey, John Wiley & Sons, Inc., 2012, 552 p. Available: https://doi.org/10.1002/9781118122631

[23] V.V. Romanuke, "Finite approximation of zero-sum games played in staircase-function continuous spaces", in *KPI Science News*, 2021, no. 4, pp. 19–38. Available: https://doi.org/10.20535/kpisn.2021.4.242769

[24] V.V. Romanuke, "Adaptive finite approximation of continuous noncooperative games", in *Journal of Automation and Information Sciences*, 2020, vol. 52, iss. 10, pp. 31–41. Available: https://doi.org/10.1615/JAutomatInfScien.v52.i10.20

В. В. Романюк

СКІНЧЕННА АПРОКСИМАЦІЯ БЕЗКОАЛІЦІЙНИХ ІГОР ДВОХ ОСІБ, ЩО РОЗІГРУЮТЬСЯ У НЕПЕРЕРВНИХ ПРОСТОРАХ СХОДИНКОВИХ ФУНКЦІЙ

Проблематика. Існує відомий спосіб апроксимації неперервних безкоаліційних ігор двох осіб, де наближений розв'язок (ситуація рівноваги) вважають прийнятним, якщо він змінюється мінімально за мінімальної зміни кроку дискретизації. Однак цей метод не можна прямо застосувати до гри двох осіб, що розігрується зі стратегіями у формі сходинкових функцій. Крім того, слід брати до уваги незалежність вибору гравцем кроку дискретизації.

Мета дослідження. Мета полягає у тому, щоб розробити метод скінченної апроксимації ігор двох осіб, які розігруються у неперервних просторах сходинкових функцій, беручи до уваги, що гравці, ймовірно, дискретизують множини своїх чистих стратегій самостійно.

Методика реалізації. Для досягнення зазначеної мети формалізується гра двох осіб, в якій стратегії гравців є сходинковими функціями часу. У такій грі множина чистих стратегій гравця є континуумом сходинкових функцій часу, і час вважають дискретним. Умови дискретизації множини можливих значень чистої стратегії гравця викладаються так, що гра стає визначеною на добутку скінченних просторів сходинкових функцій. Загалом крок дискретизації у кожного гравця різний, і розподіл вибіркових точок (значень функції-стратегії) неоднорідний.

Результати дослідження. Подано метод скінченної апроксимації ігор двох осіб, які розігруються у неперервних просторах сходинкових функцій. Метод полягає у нерегулярній дискретизації множини значень чистої стратегії гравця, знаходженні найкращих ситуацій рівноваги у "менших" біматричних іграх, кожна з яких визначена на підінтервалі, де значення чистої стратегії є постійним, й укладанні цих рівноважних ситуацій, якщо вони є узгодженими. Уклад рівноваг у "менших" біматричних іграх є наближеною рівновагою у вихідній сходинковій грі. Досліджується (слабка) узгодженість наближеної рівноваги тим, наскільки змінюється виграш та рівноважна ситуація, коли щільність дискретизації мінімально збільшується трьома способами: лише приріст у першого гравця, лише приріст у другого гравця, приріст в обох гравців. Узгодженість розкладається на узгодженість виграшів, узгодженість потужності спектра рівноважної стратегії, узгодженість щільності дискретизації рівноважної стратегії та узгодженість спектральних імовірностей. Із практичної точки зору доцільно розглядати релаксовану узгодженість виграшів.

Висновки. Запропонований метод скінченної апроксимації сходинкових ігор двох осіб полягає у незалежних дискретизаціях, розв'язуванні "менших" біматричних ігор за прийнятний проміжок часу та укладенні їхніх розв'язків, якщо вони є узгодженими.

Скінченне наближення вважають прийнятним, якщо принаймні відповідна наближена (укладена) рівновага є узгодженою за ε-виграшами.

Ключові слова: теорія ігор; функціонал виграшів; стратегія у формі сходинкової функції; біматрична гра; нерегулярна дискретизація; узгодженість наближеної рівноваги.

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