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FINITE APPROXIMATION OF ZERO-SUM GAMES PLAYED IN STAIRCASE-FUNCTION CONTINUOUS SPACES

Background. There is a known method of approximating continuous zero-sum games, wherein an approximate solution is considered acceptable if it changes minimally by changing the sampling step minimally. However, the method cannot be applied straightforwardly to a zero-sum game played with staircase-function strategies. Besides, the independence of the player's sampling step selection should be taken into account.
Objective. The objective is to develop a method of finite approximation of zero-sum games played in staircase-function continuous spaces by taking into account that the players are likely to independently sample their pure strategy sets.
Methods. To achieve the said objective, a zero-sum game, in which the players' strategies are staircase functions of time, is formalized. In such a game, the set of the player's pure strategies is a continuum of staircase functions of time, and the time is thought of as it is discrete. The conditions of sampling the set of possible values of the player's pure strategy are stated so that the game becomes defined on a product of staircase-function finite spaces. In general, the sampling step is different at each player and the distribution of the sampled points (function-strategy values) is non-uniform.
Results. A method of finite approximation of zero-sum games played in staircase-function continuous spaces is presented. The method consists in irregularly sampling the player's pure strategy value set, solving smaller-sized matrix games, each defined on a subinterval where the pure strategy value is constant, and stacking their solutions if they are consistent. The stack of the smaller-sized matrix game solutions is an approximate solution to the initial staircase game. The (weak) consistency of the approximate solution is studied by how much the payoff and optimal situation change as the sampling density minimally increases by the three ways of the sampling increment: only the first player's increment, only the second player's increment, both the players' increment. The consistency is decomposed into the payoff, optimal strategy support cardinality, optimal strategy sampling density, and support probability consistency. It is practically reasonable to consider a relaxed payoff consistency.
Conclusions. The suggested method of finite approximation of staircase zero-sum games consists in the independent samplings, solving smaller-sized matrix games in a reasonable time span, and stacking their solutions if they are consistent. The finite approximation is regarded appropriate if at least the respective approximate (stacked) solution is ε-payoff consistent.
Keywords: game theory; payoff functional; staircase-function strategy; matrix game; irregular sampling; approximate solution consistency.

Introduction

Zero-sum (also known as antagonistic) games are usually used to model processes where two sides referred to as persons or players interact in struggling for optimizing the to-be-paid-or-pay events [1], [2]. A possible action of the player called its (pure) strategy can be intended to bring a particular contribution into the interaction process in order to receive the best payoff for this player [3, 4]. The strategy can be as a simple (point) action whose duration is usually short, as well as a process consisting of an order of simple actions [3, 5].

The simplest zero-sum game is a matrix game, whichever the pure strategy complexity is. Any matrix game has optimal solutions – one, a finite number, or continuum, either in pure or mixed strategies [1, 6, 7]. A more complicated zero-sum game is the antagonistic game, in which the game payoff function (kernel) is a surface defined on a finite-dimensional compact Euclidean subspace. A simple example of the subspace is a unit square [1, 6, 7]. In such cases, opposed to matrix games, the optimal solution is not always determinable. Additional complications may arise when the surface has discontinuities [8, 9]. Moreover, zero-sum games

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defined on open or half-open subspaces (e. g., open square like in the examples from [6, 7]) may not have an optimal solution at all [1, 5, 6]. Therefore, aiming at assuredly obtaining an optimal solution, rendering a zero-sum game to a matrix one becomes a crucial task in zero-sum-game modeling. Without rendering, a zero-sum game may have an intractable optimal solution (if any), when the optimal strategy support is infinite or continuous (e. g., see the examples in [6, 7, 10, 11]). A zero-sum game, in which the player's strategy is a function (e. g., of time), is a far more complicated case. In such games, the payoff kernel must be a functional mapping every pair of functions (pure strategies of the players) into a real value [7, 8, 12, 13]. A game played with such function-strategies is rendered down to a matrix game only when each of the players possesses a finite set of one's function-strategies. Obviously, the rendering is theoretically impossible if the set of the player's strategies is infinite.

The question of rendering an infinite game to a finite one was studied in [14, 15]. The core consists in approximating the infinite game so that the approximated game would not lose the properties of the initial game. There are two fundamental conditions in the game approximation core that allow rendering a zero-sum game with strategies as functions down to a matrix game. First, a time interval, on which the pure strategy is defined, should be broken into a set of subintervals, on which the strategy could be (maybe, approximately) considered constant. The second condition requires that the set of possible values of the player's function-strategy be finite.

The first fundamental condition is the time sampling condition. It can be done according to the rules of a system to be game-modelled, where the administrator (supervisor, manager, controller, etc.) does always define (or constrain) the form of the strategies players will use [8, 10, 16, 17]. Moreover, any process is interpreted static on a sufficiently short time span. Henceforward, the time sampling condition is considered fulfilled (automatically, by default). Then the function-strategy becomes staircase. To keep the terminology simple, the respective game can be called staircase.

The second fundamental condition is imposed for the natural reason that the number of factual actions of the players (in any game) is always finite. While the players may use strategies of whichever form they want, the number of their actions has a natural limit (unless the game is everlasting; but the everlasting game is an unreal mathematical object) [6, 7, 10, 17]. Thus, the set of function-strategies used in a zero-sum game is finite anyway. Therefore, any non-everlasting zero-sum game is played as if it is a matrix game. However, the size of this matrix game depends on how each of the players has decided on discretizing (i. e., finitely approximating) one's set of function-strategy values. It does not seem that a player is likely to independently discretize the set identically to the other player's discretization.

Theoretically, the continuous game approximation is based on sampling (discretizing) either the payoff kernel or the sets of players' pure strategies. Basically, this is the same. A method of approximating continuous zero-sum games is known from [14, 15, 18]. An approximate solution is considered acceptable if it changes minimally by changing the sampling step minimally. Obviously, the independence of the player's sampling step selection should be taken into consideration. Moreover, the method cannot be applied straightforwardly to a zero-sum game played with staircase-function strategies. However, a part of the staircase game considered on a time subinterval where the players' strategies are constant can be directly approximated by the method taking into account the independence of the player's sampling step selection.

Problem statement

Issued from the impossibility of solving zero-sum games played in staircase-function continuous spaces, the objective is to develop a method of finite approximation of such games by taking into account that the players are likely to independently sample their pure strategy sets. For achieving the objective, the following six tasks are to be fulfilled:

1. To formalize a zero-sum game, in which the players' strategies are functions of time.

2. To formalize a zero-sum game, in which the players' strategies are staircase functions. In such a game, the set of the player's pure strategies is a continuum of staircase functions of time, and the time is thought of as it is discrete.

3. To state conditions of sampling the set of possible values of the player's pure strategy so that the game be defined on a product of staircase-function finite spaces. By this, the sampling step is to be different at each player. In addition, the distribution of the sampled points (function-strategy values) must not be necessarily uniform.

4. To state conditions of the appropriate finite approximation. This implies also convergence. The independence of the player's sampling step selection is to be discussed also.

5. To discuss applicability and significance of the method for the game theory. In particular, the question of how to reconcile the difference of the players' sampling step selection is to be discussed as well.

6. Make an unbiased conclusion with a clear prompt of how the research might be extended and advanced.

A zero-sum game played with strategies as functions

A zero-sum game, in which the player's pure strategy is a function of time, is defined by the following presumptions. Let each of the players use time-varying strategies defined almost everywhere on interval $[t_1; t_2]$ by $t_2 > t_1$. The first player's strategy is denoted by $x(t)$ and the second player's strategy is denoted by $y(t)$. These functions are bounded, i. e.

$$a_{\min} \le x(t) \le a_{\max} \text{ by } a_{\min} < a_{\max} \tag{1}$$

and

$$b_{\min} \le y(t) \le b_{\max} \text{ by } b_{\min} < b_{\max}. \tag{2}$$

Besides, the square of the function-strategy is presumed to be Lebesgue-integrable [19]. The sets of the players' pure strategies are

$$X = \{x(t),\, t \in [t_1; t_2],$$
$$t_1 < t_2 : a_{\min} \le x(t) \le a_{\max} \text{ by } a_{\min} < a_{\max}\} \subset \mathbb{L}_2[t_1; t_2] \tag{3}$$

and

$$Y = \{y(t),\, t \in [t_1; t_2],$$
$$t_1 < t_2 : b_{\min} \le y(t) \le b_{\max} \text{ by } b_{\min} < b_{\max}\} \subset \mathbb{L}_2[t_1; t_2], \tag{4}$$

respectively. Each of sets (3) and (4) is a rectangular functional space, in which every element is a bounded function of time by (1) and (2).

The first player's payoff in situation

$$\{x(t),\, y(t)\} \tag{5}$$

is

$$K(x(t),\, y(t)).$$

The payoff is presumed to be an integral functional:

$$K(x(t),\, y(t)) = \int_{[t_1;\, t_2]} f(x(t),\, y(t),\, t)\, d\mu(t) \tag{6}$$

with a function

$$f(x(t),\, y(t),\, t) \tag{7}$$

of $x(t)$ and $y(t)$ explicitly including time t. Therefore, the continuous zero-sum game

$$\langle \{X, Y\},\, K(x(t),\, y(t)) \rangle \tag{8}$$

is defined on product

$$X \times Y \subset \mathbb{L}_2[t_1; t_2] \times \mathbb{L}_2[t_1; t_2] \tag{9}$$

of rectangular functional spaces (3) and (4) of players' pure strategies. It is worth noting that the game continuity is defined by the continuity of spaces (3) and (4), whereas payoff functional (6) still can have discontinuities.

As it has been argued above, zero-sum game (8), in which the players' strategies are functions of time, in practical reality is played discretely during time interval $[t_1, t_2]$. The time step is the same for each of the players because it is presumed to be established either by the rules of the system game-modelled or by the administrator.

A zero-sum game with staircase-function strategies

Denote by N the number of subintervals at which the player's pure strategy is constant, where $N \in \mathbb{N} \setminus \{1\}$. Then the player's pure strategy is a staircase function having only N different values (out of, maybe, a continuum of possible values). Then there are $N - 1$ time points at which the staircase-function strategy changes or can change its value. These points are $\{\tau^{(i)}\}_{i=1}^{N-1}$, where

$$t_1 = \tau^{(0)} < \tau^{(1)} < \tau^{(2)} < \ldots < \tau^{(N-1)} < \tau^{(N)} = t_2. \tag{10}$$

Points $\{\tau^{(i)}\}_{i=0}^{N}$ are not necessarily to be equidistant, but they are the same for each of the players and they do not change as the game is repeated (a finite number of repetitions is meant — from the practical point of view).

The staircase-function strategies are right-continuous:

$$\lim_{\substack{\varepsilon>0 \\ \varepsilon\to 0}} x(\tau^{(i)} + \varepsilon) = x(\tau^{(i)}) \tag{11}$$

and

$$\lim_{\substack{\varepsilon>0 \\ \varepsilon\to 0}} y(\tau^{(i)} + \varepsilon) = y(\tau^{(i)}) \tag{12}$$

for $i = \overline{1, N-1}$, whereas (if the strategy value changes)

$$\lim_{\substack{\varepsilon>0 \\ \varepsilon\to 0}} x(\tau^{(i)} - \varepsilon) \ne x(\tau^{(i)}) \tag{13}$$

and

$$\lim_{\substack{\varepsilon>0 \\ \varepsilon\to 0}} y\left(\tau^{(i)}-\varepsilon\right) \neq y\left(\tau^{(i)}\right) \quad (14)$$

for $i = \overline{1, N-1}$. As an exception,

$$\lim_{\substack{\varepsilon>0 \\ \varepsilon\to 0}} x\left(\tau^{(N)}-\varepsilon\right) = x\left(\tau^{(N)}\right) \quad (15)$$

and

$$\lim_{\substack{\varepsilon>0 \\ \varepsilon\to 0}} y\left(\tau^{(N)}-\varepsilon\right) = y\left(\tau^{(N)}\right), \quad (16)$$

so

$$x\left(\tau^{(N-1)}\right) = x\left(\tau^{(N)}\right) \quad (17)$$

and

$$y\left(\tau^{(N-1)}\right) = y\left(\tau^{(N)}\right). \quad (18)$$

As both functions $x(t)$ and $y(t)$ are constant

$$\forall t \in \left[\tau^{(i-1)}; \tau^{(i)}\right) \text{ for } i = \overline{1, N-1} \text{ and}$$
$$\forall t \in \left[\tau^{(N-1)}; \tau^{(N)}\right],$$

then game (8) can be thought of as it is a succession of N continuous zero-sum games

$$\left\langle \left\{ [a_{\min}; a_{\max}], [b_{\min}; b_{\max}] \right\}, K(\alpha_i, \beta_i) \right\rangle \quad (19)$$

defined on product

$$[a_{\min}; a_{\max}] \times [b_{\min}; b_{\max}]$$

by

$$\alpha_i = x(t) \in [a_{\min}; a_{\max}]$$
$$\text{and } \beta_i = y(t) \in [b_{\min}; b_{\max}] \;\; \forall t \in \left[\tau^{(i-1)}; \tau^{(i)}\right)$$
$$\text{for } i = \overline{1, N-1} \text{ and } \forall t \in \left[\tau^{(N-1)}; \tau^{(N)}\right] \quad (20)$$

where the factual payoff in situation

$$\{\alpha_i, \beta_i\} \quad (21)$$

is

$$K(\alpha_i, \beta_i) = \int\limits_{\left[\tau^{(i-1)}; \tau^{(i)}\right)} f(\alpha_i, \beta_i, t)\, d\mu(t)$$
$$\forall i = \overline{1, N-1} \quad (22)$$

and

$$K(\alpha_N, \beta_N) = \int\limits_{\left[\tau^{(N-1)}; \tau^{(N)}\right]} f(\alpha_N, \beta_N, t)\, d\mu(t). \quad (23)$$

Henceforward, game (8) equivalent to the succession of N continuous zero-sum games (19) by (20)–(23) is called staircase. A pure-strategy situation in staircase game (8) is a succession of N situations

$$\left\{\{\alpha_i, \beta_i\}\right\}_{i=1}^{N} \quad (24)$$

in games (19). In staircase game (8), the set of the player's pure strategies is still a continuum of staircase functions of time, but the time is discrete. This time-discretization property, implying constant values of the players' strategies on every subinterval, allows, in addition to the succession of N continuous zero-sum games (19), decomposing staircase game (8) with respect to the (staircase) payoff.

Theorem 1. In a pure-strategy situation (5) of staircase game (8), represented as a succession of N games (19), functional (6) is re-written as a subinterval-wise sum

$$K(x(t), y(t)) = \sum_{i=1}^{N} K(\alpha_i, \beta_i) =$$
$$= \sum_{i=1}^{N-1} \int\limits_{\left[\tau^{(i-1)}; \tau^{(i)}\right)} f(\alpha_i, \beta_i, t)\, d\mu(t) +$$
$$+ \int\limits_{\left[\tau^{(N-1)}; \tau^{(N)}\right]} f(\alpha_N, \beta_N, t)\, d\mu(t). \quad (25)$$

Proof. Situation (21) is tied to half-subinterval

$$\left[\tau^{(i-1)}; \tau^{(i)}\right) \text{ by } i = \overline{1, N-1}$$

and to subinterval

$$\left[\tau^{(N-1)}; \tau^{(N)}\right] \text{ by } i = N.$$

Function (7) in this situation is some function of time t. Denote this function by $\psi_i(t)$. For situation (21) function

$$\psi_i(t) = 0 \;\; \forall t \notin \left[\tau^{(i-1)}; \tau^{(i)}\right), \quad (26)$$

and for situation

$$\{\alpha_N, \beta_N\}$$

function

$$\psi_N(t) = 0 \;\; \forall t \notin \left[\tau^{(N-1)}; \tau^{(N)}\right]. \quad (27)$$

Therefore,

$$f\left(x(t),\, y(t),\, t\right) = \sum_{i=1}^{N} \psi_i(t) \tag{28}$$

in a pure-strategy situation (5) of staircase game (8), by using (26) and (27). Consequently,

$$K\left(x(t),\, y(t)\right) = \int\limits_{[t_1;\, t_2]} f\left(x(t),\, y(t),\, t\right) d\mu(t) =$$

$$= \sum_{i=1}^{N-1} \int\limits_{\left[\tau^{(i-1)};\, \tau^{(i)}\right)} \psi_i(t) d\mu(t) + \int\limits_{\left[\tau^{(N-1)};\, \tau^{(N)}\right]} \psi_N(t) d\mu(t) =$$

$$= \sum_{i=1}^{N-1} \int\limits_{\left[\tau^{(i-1)};\, \tau^{(i)}\right)} f\left(\alpha_i,\, \beta_i,\, t\right) d\mu\left(t\right) +$$

$$+ \int\limits_{\left[\tau^{(N-1)};\, \tau^{(N)}\right]} f\left(\alpha_N,\, \beta_N,\, t\right) d\mu\left(t\right) =$$

$$= \sum_{i=1}^{N} K\left(\alpha_i,\, \beta_i\right) \tag{29}$$

in a pure-strategy situation (5) of staircase game (8).

It is noteworthy that Theorem 1 can be proved also by using the property of countable additivity of the Lebesgue integral. Theorem 1 does not provide a method of solving the staircase game. Nevertheless, it provides a fundamental decomposition of the game based on the subinterval-wise summing in (25). This decomposition allows considering and solving each game (19) separately, whereupon the solutions are stitched (stacked) together.

Why the sampling must be different and irregular

There are two main arguments for considering different sampling steps at each of the players. First, the players cannot agree on the sampling step. If even they have identical ranges of function-strategy values, an agreement is impossible due to the cooperation is excluded. Second, if a player has a wider range of one's function-strategy values then it is likely to be sampled with a greater number of points. This, however, does not mean a denser sampling.

The sampling can be non-uniform (irregular). Indeed, a player may tend to use greater or lesser values of one's function-strategy more frequently. In particular, this may lead to a denser sampling in a neighbourhood of those values. Thus, the sampling, in a generalized approach to finite approximation of zero-sum games played in staircase-function continuous spaces, must be irregular.

Sampling along the pure strategy value axis

In game (19) on subinterval i, the first player has its set $[a_{\min};\, a_{\max}]$ of pure strategies, and the second player's pure strategy set is $[b_{\min};\, b_{\max}]$. Let set $[a_{\min};\, a_{\max}]$ be sampled non-uniformly (irregularly) with M points, $M \in \mathbb{N} \setminus \{1\}$:

$$A\left(M\right) = \left\{a^{(m)}\right\}_{m=1}^{M} =$$

$$= \left\{a_{\min},\, \left\{a^{(m)}\right\}_{m=2}^{M-1},\, a_{\max}\right\} \subset \left[a_{\min};\, a_{\max}\right] \tag{30}$$

by

$$a^{(1)} = a_{\min} \text{ and } a^{(M)} = a_{\max}, \tag{31}$$

i. e., the endpoints are always included into the sampling. Similarly to this, let set $[b_{\min};\, b_{\max}]$ be sampled non-uniformly (irregularly) with J points, $J \in \mathbb{N} \setminus \{1\}$:

$$B\left(J\right) = \left\{b^{(j)}\right\}_{j=1}^{J} =$$

$$= \left\{b_{\min},\, \left\{b^{(j)}\right\}_{j=2}^{J-1},\, b_{\max}\right\} \subset \left[b_{\min};\, b_{\max}\right] \tag{32}$$

by

$$b^{(1)} = b_{\min} \text{ and } b^{(J)} = b_{\max}. \tag{33}$$

The roughest sampling is by $M = 2$ and $J = 2$, when

$$A\left(2\right) = \left\{a^{(1)},\, a^{(2)}\right\} = \left\{a_{\min},\, a_{\max}\right\}$$

and

$$B\left(2\right) = \left\{b^{(1)},\, b^{(2)}\right\} = \left\{b_{\min},\, b_{\max}\right\}.$$

If either of integers M and J is increased by 1, a new sampling must comply with the previous one. This is a requirement of the proper sampling increment.

Definition 1. Sampling

$$\Psi\left(S+1\right) = \left\{\lambda^{(s)}\right\}_{s=1}^{S+1} =$$

$$= \left\{\zeta_{\min},\, \left\{\lambda^{(s)}\right\}_{s=2}^{S},\, \zeta_{\max}\right\} \subset \left[\zeta_{\min};\, \zeta_{\max}\right] \tag{34}$$

by $\zeta_{\min} < \zeta_{\max}$ and $S \in \mathbb{N} \setminus \{1\}$ is a proper sampling increment of sampling

$$\Psi\left(S\right) = \left\{\zeta^{(s)}\right\}_{s=1}^{S} =$$

$$= \left\{\zeta_{\min},\, \left\{\zeta^{(s)}\right\}_{s=2}^{S-1},\, \zeta_{\max}\right\} \subset \left[\zeta_{\min};\, \zeta_{\max}\right] \tag{35}$$

if

$$\max_{s=\overline{1,\,S}}\left(\lambda^{(s+1)}-\lambda^{(s)}\right)<\max_{s=\overline{1,\,S-1}}\left(\zeta^{(s+1)}-\zeta^{(s)}\right), \quad (36)$$

i. e. the $S+1$ points in 1-incremented sampling (34) are selected denser than S points in sampling (35).

With the sampling by (30)—(33), the succession of N continuous games (19) by (20), (22), (23) becomes a succession of N matrix $M\times J$ games

$$\left\langle\left\{\left\{a^{(m)}\right\}_{m=1}^{M},\left\{b^{(j)}\right\}_{j=1}^{J}\right\},\mathbf{K}_i\left(M,J\right)\right\rangle \quad (37)$$

with payoff matrices

$$\mathbf{K}_i\left(M,J\right)=\left[k_{imj}\left(M,J\right)\right]_{M\times J} \quad (38)$$

whose elements are

$$k_{imj}\left(M,J\right)=\int\limits_{\left[\tau^{(i-1)};\,\tau^{(i)}\right)} f\left(a^{(m)},b^{(j)},t\right)d\mu\left(t\right) \text{ for } i=\overline{1,\,N-1} \quad (39)$$

and

$$k_{Nmj}\left(M,J\right)=\int\limits_{\left[\tau^{(N-1)};\,\tau^{(N)}\right]} f\left(a^{(m)},b^{(j)},t\right)d\mu\left(t\right). \quad (40)$$

So, if integers M and J for game (8) by (20) is somehow selected, the staircase game is represented as a succession of N matrix $M\times J$ games (37).

By sampling (30) and (32) the staircase game becomes defined on product $A(M)\times B(J)$, which becomes a product of staircase-function finite spaces by running through all $i=\overline{1,\,N}$. Thus, game (8) becomes a finite staircase game. It might be rendered to a matrix game in order to obtain a staircase solution (herein, adjective "staircase" gives a hint to the type of the game, rather than to the structure of its solution). However, there is a much easier way to solve a finite staircase game.

Theorem 2. If game (8) on product (9) by conditions (3), (4), (6) is made a staircase game as a succession of N continuous zero-sum games (19) by (20), (22), (23), whereupon it is sampled by (30) and (32), then the respective finite staircase game is always solved as a stack of successive optimal solutions of N matrix games (37) by (38)–(40).

Proof. A matrix game always has a solution, either in pure or mixed strategies. Denote by

$$\mathbf{P}_i\left(M,J\right)=\left[p_i^{(m)}\left(M,J\right)\right]_{1\times M}$$

and

$$\mathbf{Q}_i\left(M,J\right)=\left[q_i^{(j)}\left(M,J\right)\right]_{1\times J}$$

the mixed strategies of the first and second players, respectively, in matrix game (37). The respective sets of mixed strategies of the first and second players are

$$\boldsymbol{\mathcal{P}}=\left\{\mathbf{P}_i\left(M,J\right)\in\mathbb{R}^M: p_i^{(m)}\left(M,J\right)\geq 0,\ \sum_{m=1}^{M}p_i^{(m)}\left(M,J\right)=1\right\} \quad (41)$$

and

$$\boldsymbol{\mathcal{Q}}=\left\{\mathbf{Q}_i\left(M,J\right)\in\mathbb{R}^J: q_i^{(j)}\left(M,J\right)\geq 0,\ \sum_{j=1}^{J}q_i^{(j)}\left(M,J\right)=1\right\}, \quad (42)$$

so

$$\mathbf{P}_i(M,J)\in\boldsymbol{\mathcal{P}},\quad \mathbf{Q}_i(M,J)\in\boldsymbol{\mathcal{Q}},$$

and

$$\left\{\mathbf{P}_i(M,J),\,\mathbf{Q}_i(M,J)\right\} \quad (43)$$

is a situation in this game, i. e. (43) is a situation on subinterval i. Let

$$\left\{\left\{\mathbf{P}_i^*\left(M,J\right),\mathbf{Q}_i^*\left(M,J\right)\right\}\right\}_{i=1}^{N}=\left\{\left\{\left[p_i^{(m)*}\left(M,J\right)\right]_{1\times M},\left[q_i^{(j)*}\left(M,J\right)\right]_{1\times J}\right\}\right\}_{i=1}^{N} \quad (44)$$

be a set of optimal solutions of N games (37) by (39) and (40). Then

$$\max_{\mathbf{P}_i(M,J)\in\boldsymbol{\mathcal{P}}}\ \min_{\mathbf{Q}_i(M,J)\in\boldsymbol{\mathcal{Q}}}\mathbf{P}_i\left(M,J\right)\cdot\mathbf{K}_i\left(M,J\right)\cdot\left[\mathbf{Q}_i\left(M,J\right)\right]^{\mathrm{T}}=$$

$$=\max_{\mathbf{P}_i(M,J)\in\boldsymbol{\mathcal{P}}}\ \min_{\mathbf{Q}_i(M,J)\in\boldsymbol{\mathcal{Q}}}\sum_{m=1}^{M}\sum_{j=1}^{J}k_{imj}\left(M,J\right)\times p_i^{(m)}\left(M,J\right)q_i^{(j)}\left(M,J\right)=$$

$$=\max_{\mathbf{P}_i(M,J)\in\boldsymbol{\mathcal{P}}}\ \min_{\mathbf{Q}_i(M,J)\in\boldsymbol{\mathcal{Q}}}\sum_{m=1}^{M}\sum_{j=1}^{J}p_i^{(m)}\left(M,J\right)q_i^{(j)}\left(M,J\right)\times\int\limits_{\left[\tau^{(i-1)};\,\tau^{(i)}\right)} f\left(a^{(m)},b^{(j)},t\right)d\mu\left(t\right)=$$

$$= \sum_{m=1}^{M} \sum_{j=1}^{J} p_i^{(m)*}(M, J) q_i^{(j)*}(M, J) \times$$

$$\times \int\limits_{[\tau^{(i-1)};\, \tau^{(i)})} f\left(a^{(m)}, b^{(j)}, t\right) d\mu(t) =$$

$$= \mathbf{P}_i^*(M, J) \cdot \mathbf{K}_i(M, J) \cdot \left[\mathbf{Q}_i^*(M, J)\right]^{\mathrm{T}} = v_i^*(M, J) =$$

$$= \min_{\mathbf{Q}_i(M, J) \in \boldsymbol{Q}} \max_{\mathbf{P}_i(M, J) \in \boldsymbol{P}} \sum_{m=1}^{M} \sum_{j=1}^{J} p_i^{(m)}(M, J) q_i^{(j)}(M, J) \times$$

$$\times \int\limits_{[\tau^{(i-1)};\, \tau^{(i)})} f\left(a^{(m)}, b^{(j)}, t\right) d\mu(t) =$$

$$= \min_{\mathbf{Q}_i(M, J) \in \boldsymbol{Q}} \max_{\mathbf{P}_i(M, J) \in \boldsymbol{P}} \sum_{m=1}^{M} \sum_{j=1}^{J} k_{imj}(M, J) \times$$

$$\times p_i^{(m)}(M, J) q_i^{(j)}(M, J) =$$

$$= \min_{\mathbf{Q}_i(M, J) \in \boldsymbol{Q}} \max_{\mathbf{P}_i(M, J) \in \boldsymbol{P}} \mathbf{P}_i(M, J) \cdot \mathbf{K}_i(M, J) \cdot \left[\mathbf{Q}_i(M, J)\right]^{\mathrm{T}}$$

$$\forall\, i = \overline{1, N-1} \qquad (45)$$

and

$$\max_{\mathbf{P}_N(M, J) \in \boldsymbol{P}} \min_{\mathbf{Q}_N(M, J) \in \boldsymbol{Q}} \mathbf{P}_N(M, J) \cdot \mathbf{K}_N(M, J) \cdot \left[\mathbf{Q}_N(M, J)\right]^{\mathrm{T}} =$$

$$= \max_{\mathbf{P}_N(M, J) \in \boldsymbol{P}} \min_{\mathbf{Q}_N(M, J) \in \boldsymbol{Q}} \sum_{m=1}^{M} \sum_{j=1}^{J} k_{Nmj}(M, J) \times$$

$$\times p_N^{(m)}(M, J) q_N^{(j)}(M, J) =$$

$$= \max_{\mathbf{P}_N(M, J) \in \boldsymbol{P}} \min_{\mathbf{Q}_N(M, J) \in \boldsymbol{Q}} \sum_{m=1}^{M} \sum_{j=1}^{J} p_N^{(m)}(M, J) q_N^{(j)}(M, J) \times$$

$$\times \int\limits_{[\tau^{(N-1)};\, \tau^{(N)}]} f\left(a^{(m)}, b^{(j)}, t\right) d\mu(t) =$$

$$= \sum_{m=1}^{M} \sum_{j=1}^{J} p_N^{(m)*}(M, J) q_N^{(j)*}(M, J) \times$$

$$\times \int\limits_{[\tau^{(N-1)};\, \tau^{(N)}]} f\left(a^{(m)}, b^{(j)}, t\right) d\mu(t) =$$

$$= \mathbf{P}_N^*(M, J) \cdot \mathbf{K}_N(M, J) \cdot \left[\mathbf{Q}_N^*(M, J)\right]^{\mathrm{T}} = v_N^*(M, J) =$$

$$= \min_{\mathbf{Q}_N(M, J) \in \boldsymbol{Q}} \max_{\mathbf{P}_N(M, J) \in \boldsymbol{P}} \sum_{m=1}^{M} \sum_{j=1}^{J} p_N^{(m)}(M, J) q_N^{(j)}(M, J) \times$$

$$\times \int\limits_{[\tau^{(N-1)};\, \tau^{(N)}]} f\left(a^{(m)}, b^{(j)}, t\right) d\mu(t) =$$

$$= \min_{\mathbf{Q}_N(M, J) \in \boldsymbol{Q}} \max_{\mathbf{P}_N(M, J) \in \boldsymbol{P}} \sum_{m=1}^{M} \sum_{j=1}^{J} k_{Nmj}(M, J) \times$$

$$\times p_N^{(m)}(M, J) q_N^{(j)}(M, J) =$$

$$= \min_{\mathbf{Q}_N(M, J) \in \boldsymbol{Q}} \max_{\mathbf{P}_N(M, J) \in \boldsymbol{P}} \mathbf{P}_N(M, J) \times$$

$$\times \mathbf{K}_N(M, J) \cdot \left[\mathbf{Q}_N(M, J)\right]^{\mathrm{T}}. \qquad (46)$$

Using Theorem 1 allows concluding that, using a wide-sense pure-strategy symbolism,

$$\max_{x(t) \in X} \min_{y(t) \in Y} K\left(x(t), y(t)\right) =$$

$$= \sum_{i=1}^{N-1} \left(\max_{\mathbf{P}_i(M, J) \in \boldsymbol{P}} \min_{\mathbf{Q}_i(M, J) \in \boldsymbol{Q}} \sum_{m=1}^{M} \sum_{j=1}^{J} p_i^{(m)}(M, J) \times \right.$$

$$\left. \times q_i^{(j)}(M, J) \int\limits_{[\tau^{(i-1)};\, \tau^{(i)})} f\left(a^{(m)}, b^{(j)}, t\right) d\mu(t) \right) +$$

$$+ \max_{\mathbf{P}_N(M, J) \in \boldsymbol{P}} \min_{\mathbf{Q}_N(M, J) \in \boldsymbol{Q}} \sum_{m=1}^{M} \sum_{j=1}^{J} p_N^{(m)}(M, J) \times$$

$$\times q_N^{(j)}(M, J) \int\limits_{[\tau^{(N-1)};\, \tau^{(N)}]} f\left(a^{(m)}, b^{(j)}, t\right) d\mu(t) =$$

$$= \sum_{i=1}^{N-1} \sum_{m=1}^{M} \sum_{j=1}^{J} p_i^{(m)*}(M, J) q_i^{(j)*}(M, J) \times$$

$$\times \int\limits_{[\tau^{(i-1)};\, \tau^{(i)})} f\left(a^{(m)}, b^{(j)}, t\right) d\mu(t) +$$

$$+ \sum_{m=1}^{M} \sum_{j=1}^{J} p_N^{(m)*}(M, J) q_N^{(j)*}(M, J) \times$$

$$\times \int\limits_{[\tau^{(N-1)};\, \tau^{(N)}]} f\left(a^{(m)}, b^{(j)}, t\right) d\mu(t) =$$

$$= \sum_{i=1}^{N-1} \mathbf{P}_i^*(M, J) \cdot \mathbf{K}_i(M, J) \cdot \left[\mathbf{Q}_i^*(M, J)\right]^{\mathrm{T}} +$$

$$+ \mathbf{P}_N^*(M, J) \cdot \mathbf{K}_N(M, J) \cdot \left[\mathbf{Q}_N^*(M, J)\right]^{\mathrm{T}} =$$

$$= \sum_{i=1}^{N-1} v_i^*(M, J) + v_N^*(M, J) = v^*(M, J) =$$

$$= \sum_{i=1}^{N-1} \left(\min_{\mathbf{Q}_i(M,J)\in\boldsymbol{Q}} \max_{\mathbf{P}_i(M,J)\in\boldsymbol{P}} \sum_{m=1}^{M}\sum_{j=1}^{J} p_i^{(m)}(M,J) \times \right.$$

$$\left. \times q_i^{(j)}(M,J) \int\limits_{\left[\tau^{(i-1)};\, \tau^{(i)}\right)} f\left(a^{(m)}, b^{(j)}, t\right) d\mu(t) \right) +$$

$$+ \min_{\mathbf{Q}_N(M,J)\in\boldsymbol{Q}} \max_{\mathbf{P}_N(M,J)\in\boldsymbol{P}} \sum_{m=1}^{M}\sum_{j=1}^{J} p_N^{(m)}(M,J) \times$$

$$\times q_N^{(j)}(M,J) \int\limits_{\left[\tau^{(N-1)};\, \tau^{(N)}\right]} f\left(a^{(m)}, b^{(j)}, t\right) d\mu(t) =$$

$$= \min_{y(t)\in Y} \max_{x(t)\in X} K\left(x(t), y(t)\right) \tag{47}$$

and, therefore, the stack of successive solutions (44) is an optimal solution in staircase game (8) sampled by (30) and (32).

It is quite clear that these $M \times J$ matrix games are solved in parallel, without caring of the succession. The succession does matter when the solutions are stacked (stitched) together to form the staircase solution (the solution to the finite staircase game). If all N matrix games are solved in pure strategies, then stacking their solutions is fulfilled trivially. When there is at least an equilibrium in mixed strategies for a subinterval, the stacking is fulfilled as well implying that the resulting pure-mixed-strategy solution of staircase game (8) is realized successively, subinterval by subinterval, spending the same amount of time to implement both pure strategy and mixed strategy solutions (e. g., see [6, 7, 10, 18, 20, 21]).

Approximate solution consistency

The conditions of the appropriate finite approximation are stated by using the known method of obtaining the approximate solution of continuous antagonistic games on unit multidimensional cube with uniform sampling [18]. There are five items of the conditions. The requirement of the smooth sampling of the payoff kernel is inapplicable here [22].

First of all, there is an easy-to-find condition of the finite approximation appropriateness. It is about the game optimal value change, which must not change more by the proper sampling increment. Inasmuch as an increment is possible from the side of both the players, then this condition is a set of $3N$ inequalities:

$$\left|v_i^*(M,J) - v_i^*(M+1,J)\right| \le \left|v_i^*(M-1,J) - v_i^*(M,J)\right| \quad \text{for } i=\overline{1,N} \tag{48}$$

and

$$\left|v_i^*(M,J) - v_i^*(M,J+1)\right| \le \left|v_i^*(M,J-1) - v_i^*(M,J)\right| \quad \text{for } i=\overline{1,N} \tag{49}$$

and

$$\left|v_i^*(M,J) - v_i^*(M+1,J+1)\right| \le \left|v_i^*(M-1,J-1) - v_i^*(M,J)\right| \quad \text{for } i=\overline{1,N}. \tag{50}$$

Conditions (48)—(50) mean that, as the sampling density minimally increases, either from the side of the first or second player, the game optimal value change in an appropriate approximation should not grow.

Definition 2. An approximate solution (44) to staircase game (8) is called payoff-$\{M, J\}$-consistent if inequalities (48)—(50) hold. The players' optimal strategies in such a solution are called payoff-$\{M, J\}$-consistent.

The second condition is the change of the optimal strategy support cardinality. Denote the supports of the optimal strategies of the players by

$$\operatorname{supp}\mathbf{P}_i^*(M,J) = \left\{m_u\right\}_{u=1}^{U_i(M,J)} \subset \left\{m\right\}_{m=1}^{M} \tag{51}$$

by the respective support probabilities

$$\left\{p_i^{(m_u)*}(M,J)\right\}_{u=1}^{U_i(M,J)} \tag{52}$$

and

$$\operatorname{supp}\mathbf{Q}_i^*(M,J) = \left\{j_w\right\}_{w=1}^{W_i(M,J)} \subset \left\{j\right\}_{j=1}^{J} \tag{53}$$

by the respective support probabilities

$$\left\{q_i^{(j_w)*}(M,J)\right\}_{w=1}^{W_i(M,J)}. \tag{54}$$

Then $6N$ inequalities

$$U_i(M+1,J) \ge U_i(M,J) \quad \text{for } i=\overline{1,N}, \tag{55}$$

$$U_i(M,J+1) \ge U_i(M,J) \quad \text{for } i=\overline{1,N}, \tag{56}$$

$$U_i(M+1,J+1) \ge U_i(M,J) \quad \text{for } i=\overline{1,N}, \tag{57}$$

$$W_i(M+1,J) \ge W_i(M,J) \quad \text{for } i=\overline{1,N}, \tag{58}$$

$$W_i(M,J+1) \ge W_i(M,J) \quad \text{for } i=\overline{1,N}, \tag{59}$$

$$W_i(M+1,J+1) \ge W_i(M,J) \quad \text{for } i=\overline{1,N} \tag{60}$$

require that, by minimally increasing the sampling density, either from the side of the first or second player, the cardinalities of the supports not decrease.

Definition 3. An approximate solution (44) to staircase game (8) is called weakly support-cardinality-$\{M, J\}$-consistent if inequalities (55)—(60) hold. The players' optimal strategies in such a solution are called weakly support-cardinality-$\{M, J\}$-consistent.

Obviously, requirements (55)—(60) can be supplemented (strengthened) by considering a minimal decrement of the sampling density. Then another $6N$ inequalities

$$U_i(M, J) \geq U_i(M-1, J) \quad \text{for} \quad i = \overline{1, N}, \quad (61)$$

$$U_i(M, J) \geq U_i(M, J-1) \quad \text{for} \quad i = \overline{1, N}, \quad (62)$$

$$U_i(M, J) \geq U_i(M-1, J-1) \quad \text{for} \quad i = \overline{1, N}, \quad (63)$$

$$W_i(M, J) \geq W_i(M-1, J) \quad \text{for} \quad i = \overline{1, N}, \quad (64)$$

$$W_i(M, J) \geq W_i(M, J-1) \quad \text{for} \quad i = \overline{1, N}, \quad (65)$$

$$W_i(M, J) \geq W_i(M-1, J-1) \quad \text{for} \quad i = \overline{1, N} \quad (66)$$

are required.

Definition 4. An approximate solution (44) to staircase game (8) is called support-cardinality-$\{M, J\}$-consistent if inequalities (55)–(66) hold. The players' optimal strategies in such a solution are called support-cardinality-$\{M, J\}$-consistent.

As the sampling density minimally increases, the maximal gap between the support indices should not increase. Let $m_u(M, J)$ and $j_w(M, J)$ be the respective support indices corresponding to integers $\{M, J\}$ on a subinterval by (20). Then $6N$ inequalities

$$\max_{u=1,\, U_i(M+1, J)-1} \left[m_{u+1}(M+1, J) - m_u(M+1, J) \right] \leq$$
$$\leq \max_{u=1,\, U_i(M, J)-1} \left[m_{u+1}(M, J) - m_u(M, J) \right] \text{ for } i = \overline{1, N}, \quad (67)$$

$$\max_{u=1,\, U_i(M, J+1)-1} \left[m_{u+1}(M, J+1) - m_u(M, J+1) \right] \leq$$
$$\leq \max_{u=1,\, U_i(M, J)-1} \left[m_{u+1}(M, J) - m_u(M, J) \right] \text{ for } i = \overline{1, N}, \quad (68)$$

$$\max_{u=1,\, U_i(M+1, J+1)-1} \left[m_{u+1}(M+1, J+1) - m_u(M+1, J+1) \right] \leq$$
$$\leq \max_{u=1,\, U_i(M, J)-1} \left[m_{u+1}(M, J) - m_u(M, J) \right] \text{ for } i = \overline{1, N}, \quad (69)$$

$$\max_{w=1,\, W_i(M+1, J)-1} \left[j_{w+1}(M+1, J) - j_w(M+1, J) \right] \leq$$
$$\leq \max_{w=1,\, W_i(M, J)-1} \left[j_{w+1}(M, J) - j_w(M, J) \right] \text{ for } i = \overline{1, N}, \quad (70)$$

$$\max_{w=1,\, W_i(M, J+1)-1} \left[j_{w+1}(M, J+1) - j_w(M, J+1) \right] \leq$$
$$\leq \max_{w=1,\, W_i(M, J)-1} \left[j_{w+1}(M, J) - j_w(M, J) \right] \text{ for } i = \overline{1, N}, \quad (71)$$

$$\max_{w=1,\, W_i(M+1, J+1)-1} \left[j_{w+1}(M+1, J+1) - j_w(M+1, J+1) \right] \leq$$
$$\leq \max_{w=1,\, W_i(M, J)-1} \left[j_{w+1}(M, J) - j_w(M, J) \right] \text{ for } i = \overline{1, N} \quad (72)$$

are required.

Definition 5. An approximate solution (44) to staircase game (8) is called weakly sampling-density-$\{M, J\}$-consistent if inequalities (67)—(72) hold. The players' optimal strategies in such a solution are called weakly sampling-density-$\{M, J\}$-consistent.

Similarly to strengthening the weak (by Definition 3) support cardinality to that by Definition 4, requirements (67)—(72) can be strengthened by considering a minimal decrement of the sampling density. Then another $6N$ inequalities

$$\max_{u=1,\, U_i(M, J)-1} \left[m_{u+1}(M, J) - m_u(M, J) \right] \leq$$
$$\max_{u=1,\, U_i(M, J)-1} \left[m_{u+1}(M, J) - m_u(M, J \right. \text{ for } i = \overline{1, N}, \quad (73)$$

$$\max_{u=1,\, U_i(M, J)-1} \left[m_{u+1}(M, J) - m_u(M, J) \right] \leqslant$$
$$\leq \max_{u=1,\, U_i(M, J-1)-1} \left[m_{u+1}(M, J-1) - m_u(M, J-1) \right]$$
$$\text{for} \quad i = \overline{1, N}, \quad (74)$$

$$\max_{u=1,\, U_i(M, J)-1} \left[m_{u+1}(M, J) - m_u(M, J) \right] \leq$$
$$\leq \max_{u=1,\, U_i(M-1, J-1)-1} \left[m_{u+1}(M-1, J-1) - m_u(M-1, J-1) \right]$$
$$\text{for} \quad i = \overline{1, N}, \quad (75)$$

$$\max_{w=1,\, W_i(M, J)-1} \left[j_{w+1}(M, J) - j_w(M, J) \right] \leq$$
$$\leq \max_{w=1,\, W_i(M-1, J)-1} \left[j_{w+1}(M-1, J) - j_w(M-1, J) \right]$$
$$\text{for} \quad i = \overline{1, N}, \quad (76)$$

$$\max_{w=1,\, W_i(M, J)-1} \left[j_{w+1}(M, J) - j_w(M, J) \right] \leq$$
$$\leq \max_{w=1,\, W_i(M, J-1)-1} \left[j_{w+1}(M, J-1) - j_w(M, J-1) \right]$$
$$\text{for} \quad i = \overline{1, N}, \quad (77)$$

$$\max_{w=\overline{1,\,W_i(M,\,J)-1}} \left[j_{w+1}(M,\,J) - j_w(M,\,J) \right] \le$$

$$\le \max_{w=\overline{1,\,W_i(M-1,\,J-1)-1}} \left[j_{w+1}(M-1,\,J-1) - j_w(M-1,\,J-1) \right]$$

$$\text{for} \quad i=\overline{1,\,N} \tag{78}$$

are required.

Definition 6. An approximate solution (44) to staircase game (8) is called sampling-density-$\{M,\ J\}$-consistent if inequalities (67)–(78) hold. The players' optimal strategies in such a solution are called sampling-density-$\{M,\ J\}$-consistent.

Denote by $h_1(i;\ m,\ M,\ J)$ a polyline whose vertices are probabilities

$$\left\{ p_i^{(m)*}(M,\,J) \right\}_{m=1}^{M},$$

and denote by $h_2(i;\, j,\ M,\ J)$ a polyline whose vertices are probabilities

$$\left\{ q_i^{(j)*}(M,\,J) \right\}_{j=1}^{J}.$$

Then, by minimally increasing the sampling density, the "neighboring" polylines should not be farther from each other, i. e. inequalities

$$\max_{[0;\,1]} \left| h_1(i;\, m,\, M,\, J) - h_1(i;\, m,\, M+1,\, J) \right| \le$$

$$\le \max_{[0;\,1]} \left| h_1(i;\, m,\, M-1,\, J) - h_1(i;\, m,\, M,\, J) \right|$$

$$\text{for} \quad i=\overline{1,\,N}, \tag{79}$$

$$\max_{[0;\,1]} \left| h_1(i;\, m,\, M,\, J) - h_1(i;\, m,\, M,\, J+1) \right| \le$$

$$\le \max_{[0;\,1]} \left| h_1(i;\, m,\, M,\, J-1) - h_1(i;\, m,\, M,\, J) \right|$$

$$\text{for} \quad i=\overline{1,\,N}, \tag{80}$$

$$\max_{[0;\,1]} \left| h_1(i;\, m,\, M,\, J) - h_1(i;\, m,\, M+1,\, J+1) \right| \le$$

$$\le \max_{[0;\,1]} \left| h_1(i;\, m,\, M-1,\, J-1) - h_1(i;\, m,\, M,\, J) \right|$$

$$\text{for} \quad i=\overline{1,\,N}, \tag{81}$$

and

$$\max_{[0;\,1]} \left| h_2(i;\, j,\, M,\, J) - h_2(i;\, j,\, M+1,\, J) \right| \le$$

$$\le \max_{[0;\,1]} \left| h_2(i;\, j,\, M-1,\, J) - h_2(i;\, j,\, M,\, J) \right|$$

$$\text{for} \quad i=\overline{1,\,N}, \tag{82}$$

$$\max_{[0;\,1]} \left| h_2(i;\, j,\, M,\, J) - h_2(i;\, j,\, M,\, J+1) \right| \le$$

$$\le \max_{[0;\,1]} \left| h_2(i;\, j,\, M,\, J-1) - h_2(i;\, j,\, M,\, J) \right|$$

$$\text{for} \quad i=\overline{1,\,N}, \tag{83}$$

$$\max_{[0;\,1]} \left| h_2(i;\, j,\, M,\, J) - h_2(i;\, j,\, M+1,\, J+1) \right| \le$$

$$\le \max_{[0;\,1]} \left| h_2(i;\, j,\, M-1,\, J-1) - h_2(i;\, j,\, M,\, J) \right|$$

$$\text{for} \quad i=\overline{1,\,N}, \tag{84}$$

along with

$$\left\| h_1(i;\, m,\, M,\, J) - h_1(i;\, m,\, M+1,\, J) \right\| \le$$

$$\le \left\| h_1(i;\, m,\, M-1,\, J) - h_1(i;\, m,\, M,\, J) \right\|$$

$$\text{in } \mathbb{L}_2[0;\,1] \quad \text{for} \quad i=\overline{1,\,N}, \tag{85}$$

$$\left\| h_1(i;\, m,\, M,\, J) - h_1(i;\, m,\, M,\, J+1) \right\| \le$$

$$\le \left\| h_1(i;\, m,\, M,\, J-1) - h_1(i;\, m,\, M,\, J) \right\|$$

$$\text{in } \mathbb{L}_2[0;\,1] \quad \text{for} \quad i=\overline{1,\,N}, \tag{86}$$

$$\left\| h_1(i;\, m,\, M,\, J) - h_1(i;\, m,\, M+1,\, J+1) \right\| \le$$

$$\le \left\| h_1(i;\, m,\, M-1,\, J-1) - h_1(i;\, m,\, M,\, J) \right\|$$

$$\text{in } \mathbb{L}_2[0;\,1] \quad \text{for} \quad i=\overline{1,\,N}, \tag{87}$$

and

$$\left\| h_2(i;\, j,\, M,\, J) - h_2(i;\, j,\, M+1,\, J) \right\| \le$$

$$\le \left\| h_2(i;\, j,\, M-1,\, J) - h_2(i;\, j,\, M,\, J) \right\|$$

$$\text{in } \mathbb{L}_2[0;\,1] \quad \text{for} \quad i=\overline{1,\,N}, \tag{88}$$

$$\left\| h_2(i;\, j,\, M,\, J) - h_2(i;\, j,\, M,\, J+1) \right\| \le$$

$$\le \left\| h_2(i;\, j,\, M,\, J-1) - h_2(i;\, j,\, M,\, J) \right\|$$

$$\text{in } \mathbb{L}_2[0;\,1] \quad \text{for} \quad i=\overline{1,\,N}, \tag{89}$$

$$\left\| h_2(i;\, j,\, M,\, J) - h_2(i;\, j,\, M+1,\, J+1) \right\| \le$$

$$\le \left\| h_2(i;\, j,\, M-1,\, J-1) - h_2(i;\, j,\, M,\, J) \right\|$$

$$\text{in } \mathbb{L}_2[0;\,1] \quad \text{for} \quad i=\overline{1,\,N} \tag{90}$$

are required.

Definition 7. An approximate solution (44) to staircase game (8) is called probability-$\{M,\ J\}$-consistent if inequalities (79)–(90) hold. The players' optimal strategies in such a solution are called probability-$\{M,\ J\}$-consistent.

The solution consistency by each of Definitions 3—7 implies that both the players' optimal strategies are consistent as well. Nevertheless, it is not worth cancelling the player's optimal strategy consistency when for the other player the consistency conditions do not hold. Thus, a player's optimal strategy may be consistent while an optimal strategy of the other player is not consistent. For instance, if inequalities (55)—(57) hold, but inequalities (58)—(60) do not hold (at least one of those $3N$ inequalities is violated) and thus the second player's optimal strategy is not weakly support-cardinality-$\{M, J\}$-consistent, the first player's optimal strategy is weakly support-cardinality-$\{M, J\}$-consistent. If inequalities (48)—(50), (55)—(60), (67)—(72), (79)—(90) hold for some i, then matrix game (37), assigned to the subinterval between $\tau^{(i-1)}$ and $\tau^{(i)}$, has a weakly consistent approximate solution to the corresponding continuous game (19) by (20), (22), (23). On this basis, the weak consistency of an approximate solution to a staircase game (8) is formulated.

Definition 8. The stack of successive solutions (44) is called a weakly $\{M, J\}$-consistent approximate solution of game (8) on product (9) by conditions (3), (4), (6) if inequalities (48)–(50), (55)–(60), (67)–(72), (79)–(90) hold. The players' optimal strategies in such a solution are called weakly $\{M, J\}$-consistent.

Once again, if, say, the second player's optimal strategy is not weakly $\{M, J\}$-consistent (at least one of the respective inequalities in Definition 8 for the second player is violated), it does not mean that the first player's optimal strategy is not weakly $\{M, J\}$-consistent also. If, in this example, inequalities (48)—(50), (55)—(57), (67)—(69), (79)—(81), (85)—(87) do hold, then the first player possesses a weakly $\{M, J\}$-consistent optimal strategy, regardless of the second player's weak consistency. Similarly to strengthening Definitions 3 and 5, the weak consistency can be strengthened by adding the requirements with inequalities (61)—(66) and (73)—(78).

Definition 9. The stack of successive solutions (44) is called an $\{M, J\}$-consistent approximate solution of game (8) on product (9) by conditions (3), (4), (6) if inequalities (48)–(50) and (55)–(90) hold. The players' optimal strategies in such a solution are called $\{M, J\}$-consistent.

The approximate solution consistency theoretically proposes a better approximation than the weak consistency. The weak consistency notion by Definition 8 may be thought of as it is decomposed by Definitions 2, 3, 5, 7. Thus, the consistency notion by Definition 9 is decomposed into Definitions 2, 4, 6, 7. Even if an approximate solution is not weakly consistent, it may be, e. g., payoff-consistent. The payoff consistency is checked the easiest and fastest. A payoff-consistent solution can be sufficient to accept it as an appropriate approximate solution [1, 6, 14, 18, 22]. However, if a one of $3N$ inequalities (48)—(50) is violated, even this type of consistency does not work. Meanwhile, the violation may be induced by a very small growth of the payoff change. Therefore, it is useful and practically reasonable to consider the payoff consistency adding a relaxation to inequalities (48)—(50).

Definition 10. An approximate solution (44) to staircase game (8) is called e-payoff-{M, J}-consistent if inequalities

$$\left|v_i^*(M,J)-v_i^*(M+1,J)\right|-\\ -\varepsilon\le\left|v_i^*(M-1,J)-v_i^*(M,J)\right|\\ \text{by some } \varepsilon>0 \text{ for } i=\overline{1,N} \tag{91}$$

and

$$\left|v_i^*(M,J)-v_i^*(M,J+1)\right|-\\ -\varepsilon\le\left|v_i^*(M,J-1)-v_i^*(M,J)\right|\\ \text{by some } \varepsilon>0 \text{ for } i=\overline{1,N} \tag{92}$$

and

$$\left|v_i^*(M,J)-v_i^*(M+1,J+1)\right|\\ -\varepsilon\le\left|v_i^*(M-1,J-1)-v_i^*(M,J)\right|\\ \text{by some } \varepsilon>0 \text{ for } i=\overline{1,N} \tag{93}$$

hold. The players' optimal strategies in such a solution are called ε-payoff-$\{M, J\}$-consistent.

To ascertain whether the stack of successive solutions (44) is weakly consistent or not, the seven bunches of N matrix games (37) should be solved, where the sampling density is defined by integers

$$\{M-1,J-1\},\ \{M-1,J\},\ \{M,J-1\},\ \{M,J\},\\ \{M+1,J\},\ \{M,J+1\},\ \{M+1,J+1\}.$$

It is worth noting once again that the players select their respective integers M and J independently and, moreover, the sampling by an integer S means that those $S-2$ points within an open interval can be chosen in any way, not necessarily to be uniformly distributed through the interval. Only the requirement of the proper sampling increment (by Definition 1) is followed. Nevertheless, the consistency meant by some sampling density integers $\{M, J\}$ does not guarantee that both the players will select such sampling density. Moreover, it is hard to find a continuous zero-sum game, for which a consistent approximate solution could be determined at appropriately small integers M and J. However, it is quite naturally to expect that, as they are increased (i. e., the sampling is made denser), the approximate solutions must converge to the solution of staircase game (8). Besides, the approximate solutions must become "more" consistent, which means that more inequalities of the bunch of inequalities (48)–(50), (55)–(90) must hold.

A visual exemplification

To visually exemplify how a zero-sum staircase game is approximated by using the approximate solution consistency along with reconciling the difference of the players' sampling step selection, consider a case in which $t \in [0.4\pi; 1.6\pi]$, the set of pure strategies of the first player is

$$X = \{x(t),\, t \in [0.4\pi; 1.6\pi] : 5 \leq x(t) \leq 9\} \subset \subset \mathbb{L}_2 [0.4\pi; 1.6\pi] \quad (94)$$

and the set of pure strategies of the second player is

$$Y = \{y(t),\, t \in [0.4\pi; 1.6\pi] : 2 \leq y(t) \leq 8\} \subset \subset \mathbb{L}_2 [0.4\pi; 1.6\pi], \quad (95)$$

where each of the players is allowed to change its pure strategy value at time points

$$\{\tau^{(i)}\}_{i=1}^{5} = \{0.4\pi + 0.2\pi i\}_{i=1}^{5}.$$

The payoff functional is

$$K(x(t), y(t)) = \int\limits_{[0.4\pi;\, 1.6\pi]} \sin^2\left(0.25xt + \frac{\pi}{11}\right) \times \times \sin\left(0.53yt - \frac{7\pi}{8}\right) e^{-0.02xt} d\mu(t). \quad (96)$$

So, each of the players possesses 6-subinterval staircase function-strategies defined on interval $[0.4\pi; 1.6\pi]$. Hence, the zero-sum staircase game is represented as a succession of 6 zero-sum games (19)

$$\langle \{[5; 9], [2; 8]\},\, K(\alpha_i, \beta_i) \rangle \quad (97)$$

by

$$\alpha_i = x(t) \in [5; 9] \text{ and } \beta_i = y(t) \in [2; 8]$$

$$\forall\, t \in [0.2\pi + 0.2\pi i;\, 0.4\pi + 0.2\pi i)$$
$$\text{for } i = \overline{1, 5} \text{ and } \forall\, t \in [1.4\pi; 1.6\pi], \quad (98)$$

where the factual payoff in situation (21) is

$$K(\alpha_i, \beta_i) = \int\limits_{[0.2\pi+0.2\pi i;\, 0.4\pi+0.2\pi i)} \sin^2\left(0.25\alpha_i t + \frac{\pi}{11}\right) \times \times \sin\left(0.53\beta_i t - \frac{7\pi}{8}\right) e^{-0.02\alpha_i t} d\mu(t) \quad \forall\, i = \overline{1, 5} \quad (99)$$

and

$$K(\alpha_6, \beta_6) = \int\limits_{[1.4\pi;\, 1.6\pi]} \sin^2\left(0.25\alpha_6 t + \frac{\pi}{11}\right) \times \times \sin\left(0.53\beta_6 t - \frac{7\pi}{8}\right) e^{-0.02\alpha_6 t} d\mu(t). \quad (100)$$

Payoff functional (96) on each subinterval of set

$$\left\{ \{[0.2\pi + 0.2\pi i;\, 0.4\pi + 0.2\pi i)\}_{i=1}^{5},\, [1.4\pi; 1.6\pi] \right\} \quad (101)$$

is shown in Fig. 1.

The irregularity (non-uniformity) in the sampling is modelled as follows:

$$a_0^{(m)} = 5 + \frac{4m - 4}{M - 1} \text{ and } a^{(m)} = a_0^{(m)} + \frac{\xi_1}{M}$$
$$\text{for } m = \overline{2, M - 1} \quad (102)$$

by $a^{(1)} = 5$, $a^{(M)} = 9$, and

$$b_0^{(j)} = 2 + \frac{6j - 6}{J - 1} \text{ and } b^{(j)} = b_0^{(j)} + \frac{\xi_2}{J}$$
$$\text{for } j = \overline{2, J - 1} \quad (103)$$

by $b^{(1)} = 2$, $b^{(J)} = 8$, where ξ_1 and ξ_2 are values of two independent random variables distributed normally with zero mean and unit variance. The values resulting from (102) and (103) are sorted in ascending order, whereupon they are checked whether (30) and (32) are true. When either integer M or J is increased by 1, samplings (30) and (32) are checked whether they satisfy the proper sampling increment by Definition 1, i. e. whether inequality (36) holds for samplings (35) and (34).

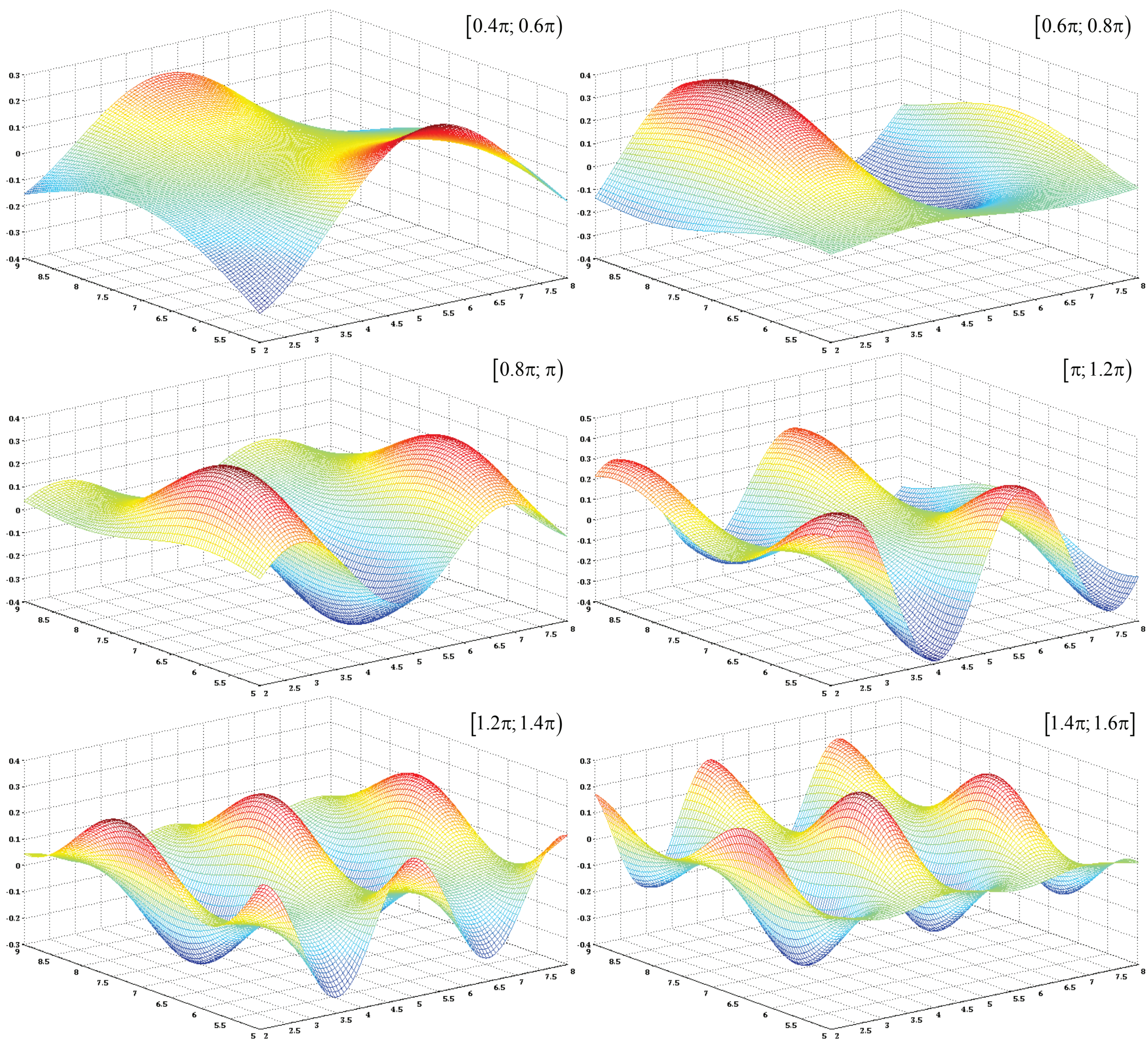

Fig. 1. The payoff kernels (99), (100) on the 6 subintervals of set (101)

Thus, 6 matrix games (37) with payoff matrices (38) are formed from 6 zero-sum games (97), where

$$k_{imj}(M, J) = \int\limits_{[0.2\pi+0.2\pi i;\, 0.4\pi+0.2\pi i)} \sin^2\left(0.25a^{(m)}t + \frac{\pi}{11}\right) \times \sin\left(0.53b^{(j)}t - \frac{7\pi}{8}\right) e^{-0.02a^{(m)}t} d\mu(t) \quad \text{for } i = \overline{1, 5} \quad (104)$$

and

$$k_{6mj}(M, J) = \int\limits_{[1.4\pi;\, 1.6\pi]} \sin^2\left(0.25a^{(m)}t + \frac{\pi}{11}\right) \times \sin\left(0.53b^{(j)}t - \frac{7\pi}{8}\right) e^{-0.02a^{(m)}t} d\mu(t). \quad (105)$$

Although the subinterval length in (104) and (105) does not change, every subinterval has its "own" matrix game due to time variable t is explicitly included into the functions under the integral. This means that, as time goes by, the players develop their actions subinterval by subinterval.

Surely, the solutions of these games (and the solution of the initial staircase game) badly depend on the sampling. Subinterval-wise optimal strategies of the players by the sampling for every $M = \overline{3, 10}$ and $J = \overline{3, 10}$ are shown in Fig. 2 in a bunch (they are presented indistinguishable). It is well seen that as the sampling density changes at such a relatively wide range of small sampling integers M and J, the player's optimal strategy (in every subinterval game, let alone the stacked optimal strategy on interval [0.4π; 1.6π]) badly varies. The only exception is the first subinterval, on which the second player's optimal strategy being the pure one does not vary at all. The first player has only pure optimal strategies on this subinterval as well. The first player's payoff $v_i^*(M, J)$ (at the end of the i-th subinterval) and the payoff cumulative sum

$$v^{(n)*}(M, J) = \sum_{i=1}^{n} v_i^*(M, J) \text{ by } n = \overline{1, 6} \quad (106)$$

are also badly scattered (Fig. 3), where, according to (106), $v^{(6)*}(M, J)$ is the optimal value in this staircase game, i. e.

$$v^*(M, J) = v^{(6)*}(M, J). \quad (107)$$

The only exception is that payoffs $v_6^*(M, J)$ received on reaching the final time point (at the end of the sixth subinterval) are almost converged (seen as dots), unlike optimal values (107) being scattered the worst (seen as circles).

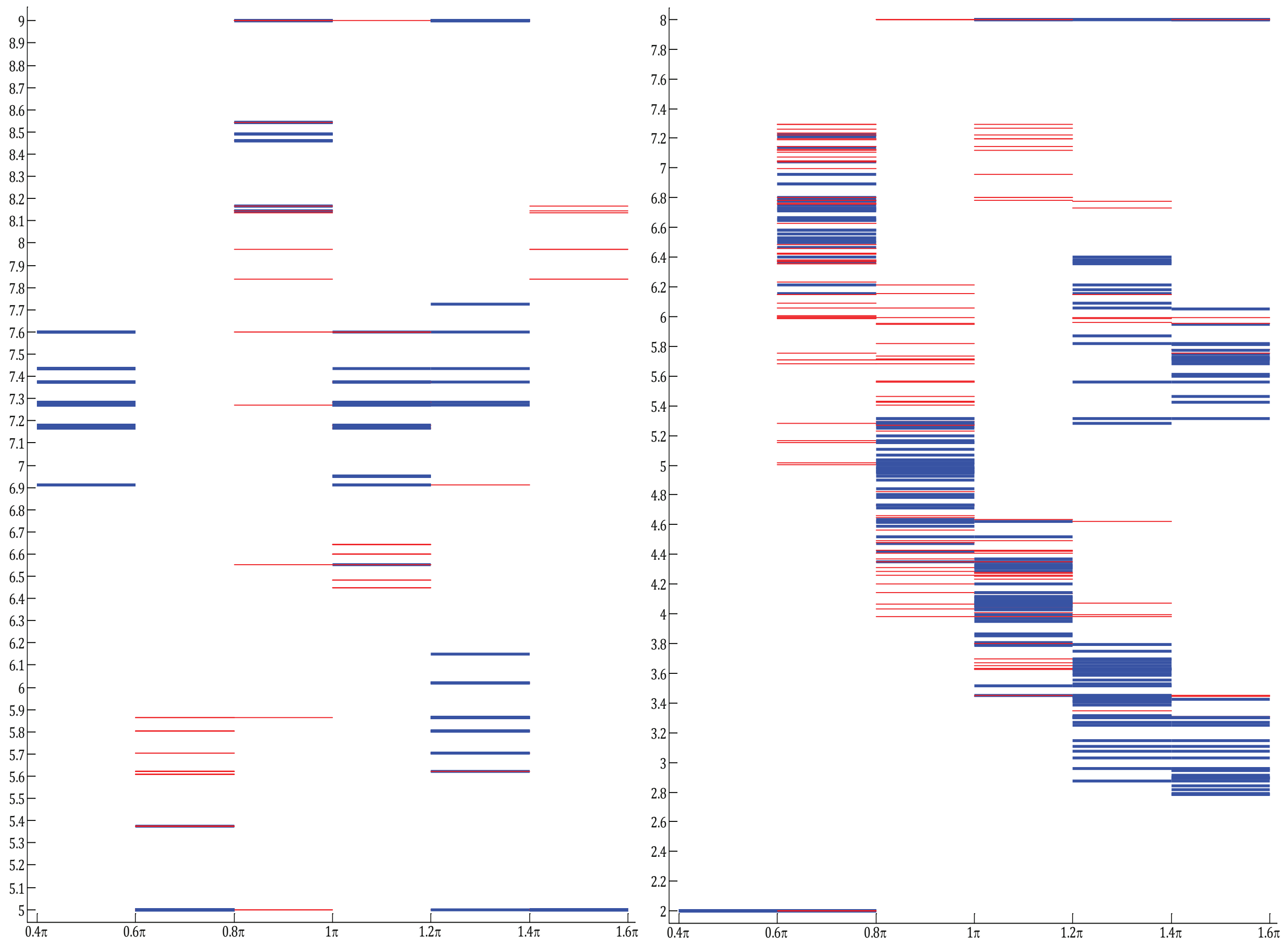

Fig. 2. An indistinguishable bunch of subinterval-wise optimal strategies of the first (left) and second (right) players by $M = \overline{3, 10}$ and $J = \overline{3, 10}$ (here and further below the optimal pure strategy is represented by thicker line, pure strategies from the mixed optimal strategy support are represented by thinner lines)

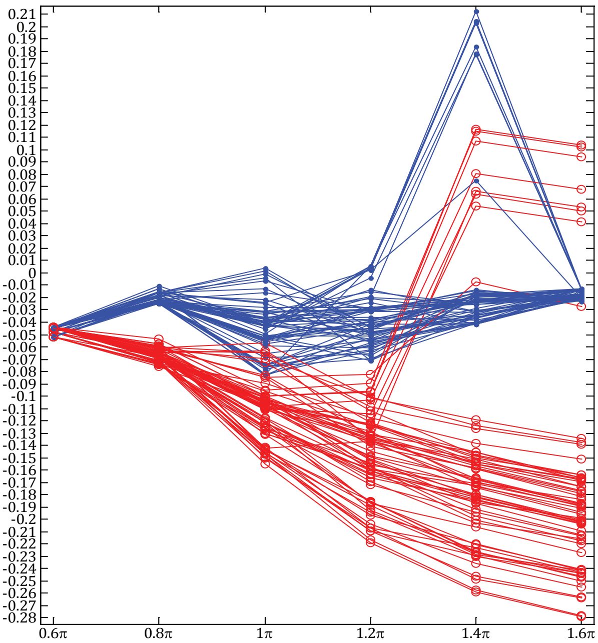

Fig. 3. An indistinguishable bunch of the first player’s payoffs at the end of every subinterval (dots) and their cumulative sum (circles) by $M = \overline{3, 10}$ and $J = \overline{3, 10}$

Nevertheless, as the sampling density is further increased, mixed optimal strategies become more “condensed” (Fig. 4), where game (97) by (98)–(100) is still solved in pure strategies on the first subinterval. The pure optimal strategy of the second player on $[0.4\pi; 0.6\pi)$ is unchangeable (it is unchangeable, whichever the sampling is). Moreover, the payoffs “condense” also (Fig. 5): the subinterval payoffs run into a distinct polyline, and their cumulative sum runs into a polyline as well, although some scattering of optimal values (107) is still seen.

It is noteworthy that the players’ optimal strategies are ε-payoff-{M, J}-consistent just for

$$\varepsilon = 0.304 \cdot \left| v_i^* \left(M, J \right) \right| \quad \text{at} \quad i = \overline{1, 6}$$

by every

$$M = \overline{15, 20} \text{ and } J = \overline{15, 20}.$$

This is an evidence of that the solution convergence is not enough.

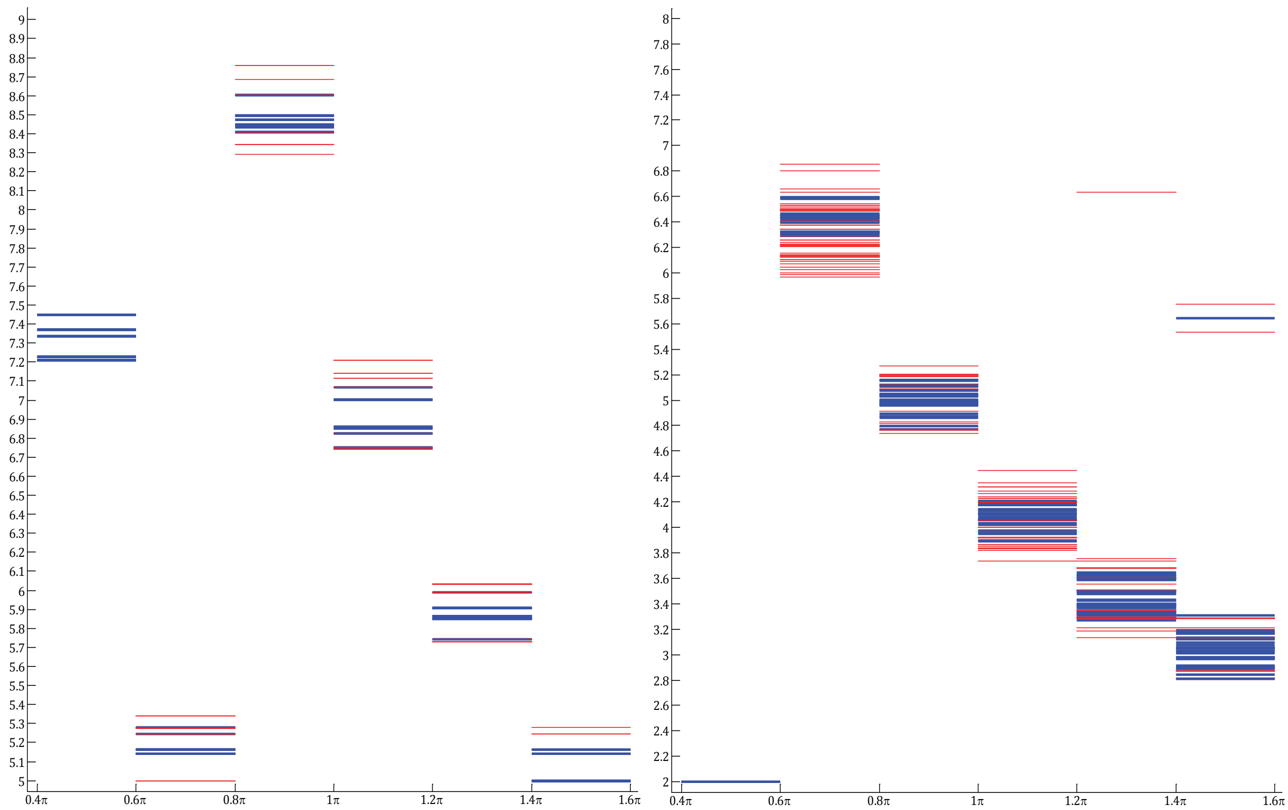

Fig. 4. A bunch of subinterval-wise optimal strategies of the first (left) and second (right) players by $M = \overline{15, 20}$ and $J = \overline{15, 20}$

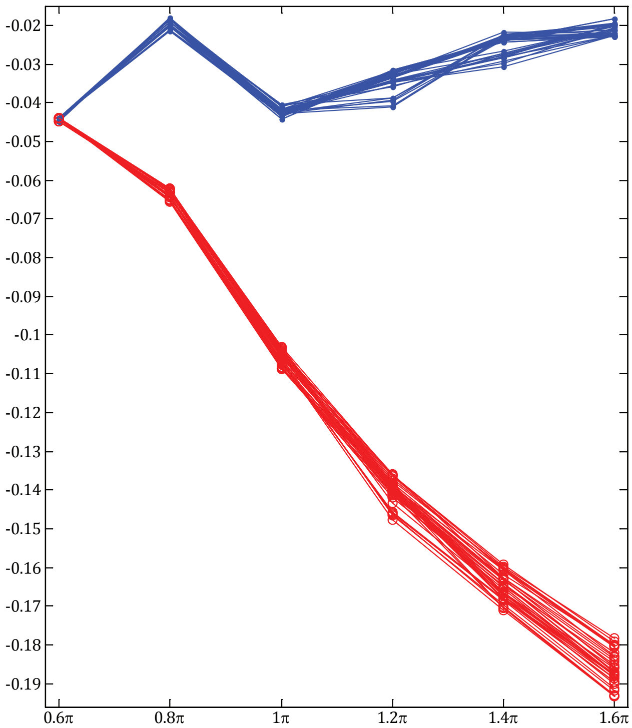

Fig. 5. A bunch of the first player's payoffs at the end of every subinterval (dots) and their cumulative sum (circles) by $M = \overline{15, 20}$ and $J = \overline{15, 20}$

Fig. 6 presenting mixed optimal strategies by $M = \overline{25, 30}$ and $J = \overline{25, 30}$ can be easily compared to Fig. 4. The matter is that, along with the first player's pure optimal strategies on subinterval $[0.4\pi; 0.6\pi)$, the player's mixed optimal strategies on subintervals

$$\left\{\left\{[0.2\pi + 0.2\pi i;\ 0.4\pi + 0.2\pi i)\right\}_{i=2}^{5},\right.$$

$$\left.[1.4\pi;\ 1.6\pi]\right\} \qquad (108)$$

do really converge to the solution of the staircase game. The comparison of more "condensed" payoffs in Fig. 7 to Fig. 5 allows concluding the same. Moreover, here the players' optimal strategies are ε-payoff-$\{M, J\}$-consistent for

$$\varepsilon = 0.159 \cdot \left|v_i^*(M, J)\right| \quad \text{at} \quad i = \overline{1, 6}$$

by every

$$M = \overline{25, 30} \ \text{ and } \ J = \overline{25, 30}$$

additionally supporting the said.

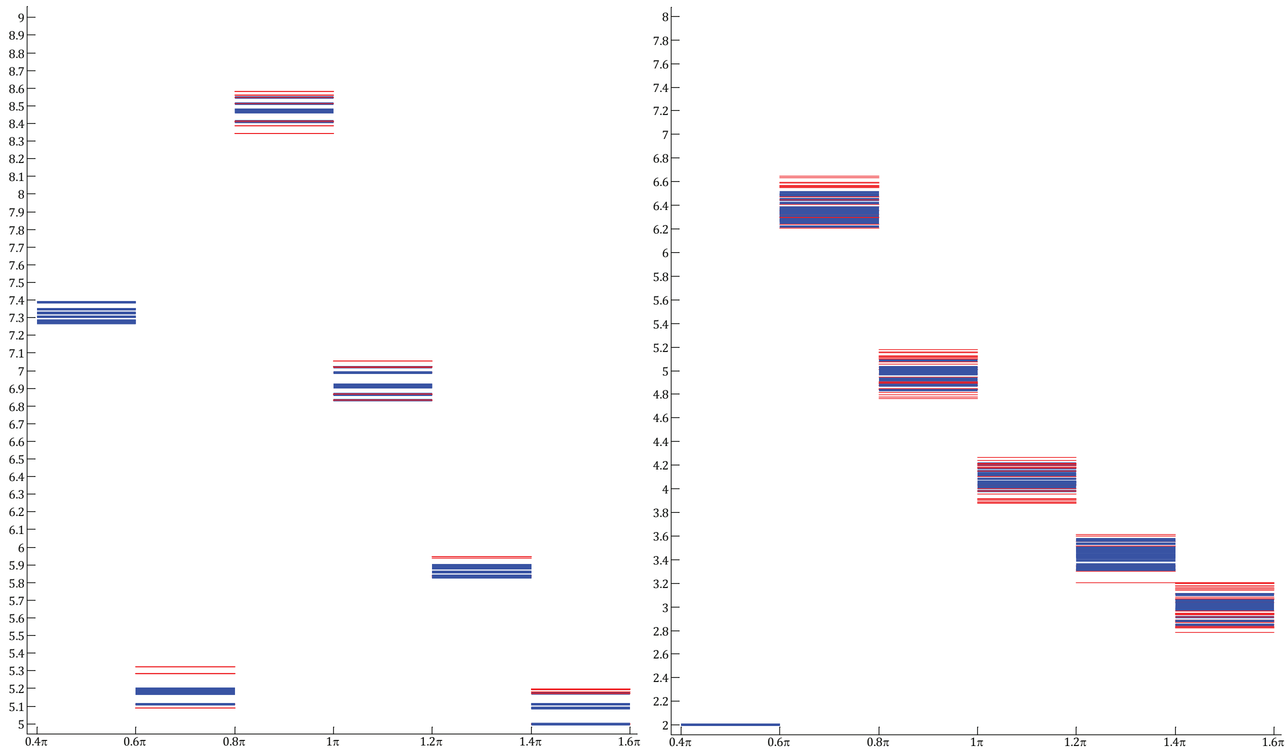

Fig. 6. A bunch of subinterval-wise optimal strategies of the first (left) and second (right) players by $M = \overline{25, 30}$ and $J = \overline{25, 30}$

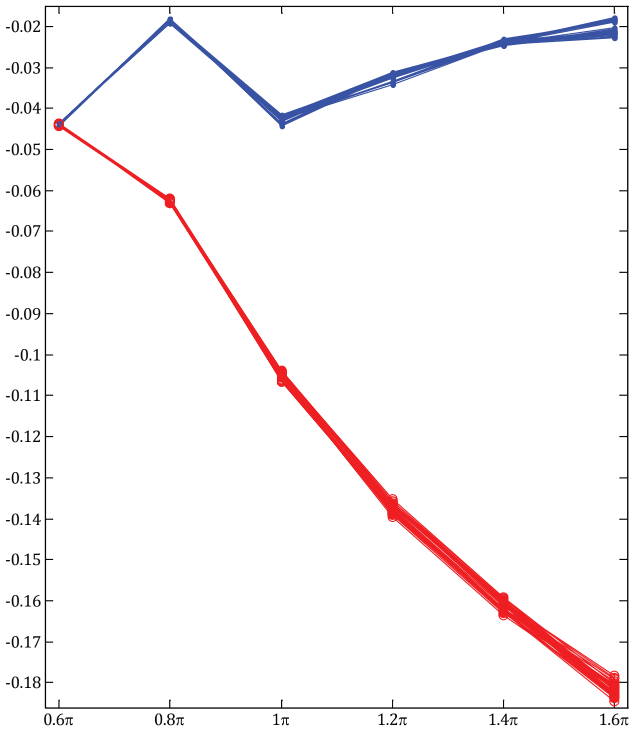

Fig. 7. A bunch of the first player's payoffs at the end of every subinterval (dots) and their cumulative sum (circles) by $M = \overline{25, 30}$ and $J = \overline{25, 30}$

Further thickening the samplings does not change the result much. Along with the first player's pure optimal strategies on subinterval $[0.4\pi; 0.6\pi)$, the scattering of mixed optimal strategies on subintervals (108) by $M = \overline{31, 40}$ and $J = \overline{31, 40}$ (Fig. 8) is slightly less than that in Fig. 6. The "condensation" of payoffs in Figures 7 and 9 are nearly the same.

Although the solution convergence is apparent, the players' optimal strategies are ε-payoff-$\{M, J\}$-consistent for

$$\varepsilon = 0.202 \cdot \left| v_i^* (M, J) \right| \quad \text{at} \quad i = \overline{1, 6}$$

by every

$$M = \overline{32, 39} \text{ and } J = \overline{32, 39}.$$

This is an evidence of that the solution convergence reaches its saturation, and further thickening the samplings will not improve the solution approximation nor improve the consistency. Therefore, the approximate solution to the zero-sum staircase game by (94)—(96) and (97)—(100) can be accepted by the independent sampling at both players' with the integers between 25 and 30 (of course, not necessarily identical).

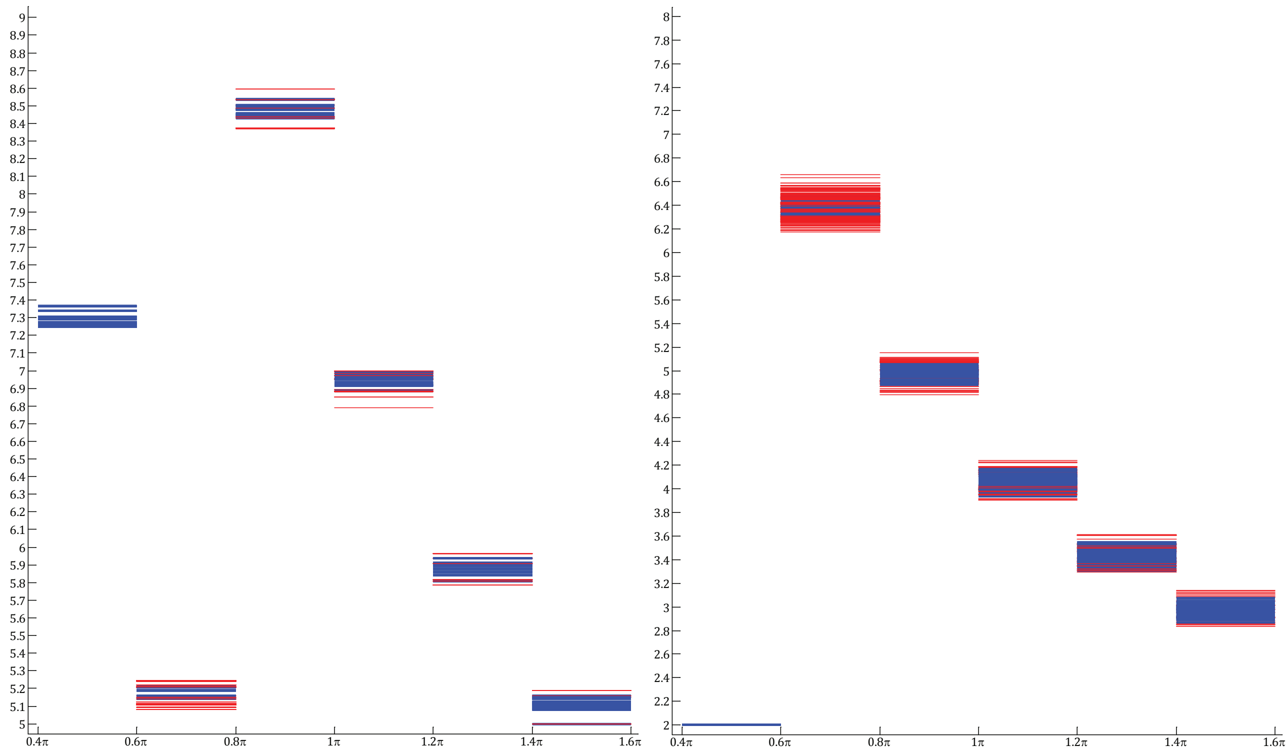

Fig. 8. A bunch of subinterval-wise optimal strategies of the first (left) and second (right) players by $M = \overline{31, 40}$ and $J = \overline{31, 40}$

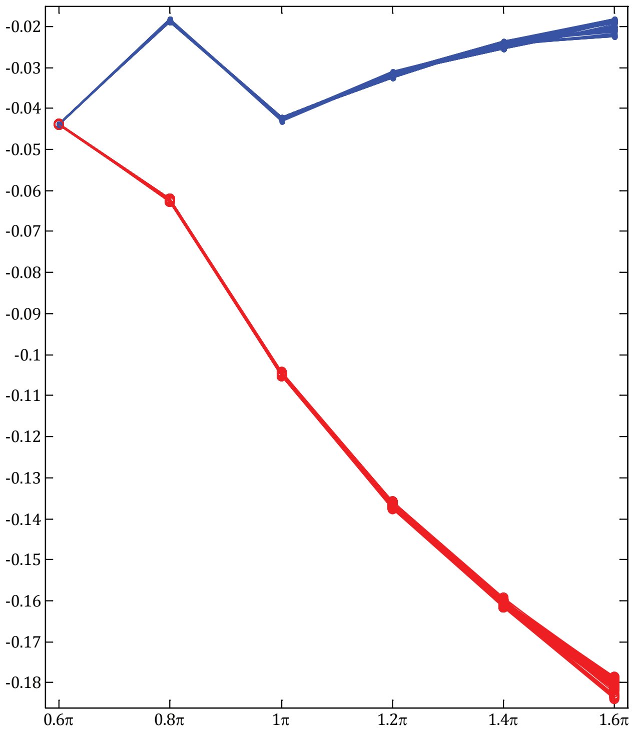

Fig. 9. A bunch of the first player's payoffs at the end of every subinterval (dots) and their cumulative sum (circles) by $M = \overline{31, 40}$ and $J = \overline{31, 40}$

Discussion of the contribution

Clearly, it would be commonly intractable to straightforwardly solve the sampled staircase game, without considering each subinterval matrix game separately. For instance, by sampling the exemplified game, where each of the players uses 6-subinterval staircase function-strategies, with, say, $M = 25$ and $J = 30$, the resulting $25^6 \times 30^6$ matrix game (in which, e. g., the second player has 729 million pure strategies!) cannot be solved in a reasonable time span. Therefore, solving subinterval matrix games (which are obviously "smaller") separately and then stacking their solutions is a far more efficient way to obtain an approximate solution of the initial staircase game. The applicability of this method may be limited to the subinterval matrix game size. For instance, the computation time has an exponentially-increasing dependence on the size of the square matrix. Solving matrix games, in which each of the players has at least a few hundred pure strategies, may be time-consuming in applications requiring fast updates of the solution (when the structure of the initial staircase game changes itself).

The (weak) consistency of an approximate solution is a criterion of its acceptability. However, a (weakly) consistent approximate solution may not exist at appropriately small (tractable) M and J. So, the consistency decomposition into parts by Definitions 2—7 and particularly isolating an ε-payoff consistency by Definition 10 is justified and practically applicable. There are still many open questions, though. First, it is not proved that limits

$$\lim_{M \to \infty,\, J \to \infty} v_i^*(M, J) \quad \forall\, i = \overline{1, N} \qquad (109)$$

exist and they are equal to the respective optimal values of the subinterval continuous games. Second, if limits (109) exist, it is not proved that this is followed by that any approximate solution (44) is ε-payoff-$\{M, J\}$-consistent for any $M \geq M_*$ and $J \geq J_*$ ($M_* \in \mathbb{N} \setminus \{1\}$, $J_* \in \mathbb{N} \setminus \{1\}$). The inter-influence among the consistency decomposition parts by Definitions 2—7 is also uncertain yet.

The question of a possible reconciliation of the difference of the players' sampling step selection is indeed that hard. The players can select their samplings simultaneously but not identically. Even if the ranges of function-strategy values are identical and sampling integers M and J are the same (i. e., $M = J$), implying the uniform samplings, a player's sampling may differ from the other player's sampling due to eventual inaccuracies in selecting points, as it has been modelled by (102) and (103) with using normal "noise" in the point selection. However, at sufficiently great sampling integers M and J, not necessarily equal, significant changes in M and J are expected not to influence the approximate solution much. Just like in the above-considered example, the player's optimal strategies converge subinterval-wise and the resulting staircase strategy appears to be an acceptable approximate optimal strategy in the initial staircase game (see Figures 6 and 8). Such a conclusion is made easier by the payoff convergence (see Figures 5, 7 and 9).

Therefore, the presented method is a significant contribution to the antagonistic game theory and its finite approximation supplement. It allows approximately solving zero-sum games with staircase-function strategies in a far simpler manner regardless of the fact that the players may sample their sets of function-strategy values differently [18, 22]. Once the (weak) consistency is confirmed (the respective approximate solution should be at least ε-payoff consistent by Definition 10), the approximate pure-mixed-strategy solution (like those ones of staircase strategies in Figures 6, 8) can be easily implemented and practiced [5, 7, 10, 11, 15, 18, 20].

Conclusion

A zero-sum game played in staircase-function continuous spaces is approximated to a matrix game by sampling the players' pure strategy value sets. Each set is irregularly sampled in its own way so that the resulting samplings may be of different cardinalities and varying densities. While sampled, the requirement of the proper sampling increment (by Definition 1) must be followed – the $S+1$ points in a 1-incremented sampling must be selected denser than S points.

Owing to Theorem 2, the solution of the matrix game is obtained by stacking the solutions of the "smaller" matrix games, each defined on a subinterval where the pure strategy value is constant. The stack of the "smaller" matrix game solutions is an approximate solution to the initial staircase game. The (weak) consistency of the approximate solution is studied by how much the payoff and optimal situation change as the sampling density minimally increases by the three ways of the sampling increment: only the first player's increment, only the second player's increment, both the players' increment. Thus, the consistency, equivalent to the approximate solution acceptability, is decomposed into the payoff (Definition 2), optimal strategy support cardinality (Definitions 3 and 4), optimal strategy sampling density (Definitions 5 and 6), and support probability consistency (Definition 7).

The most important parts are the payoff consistency and optimal strategy support cardinality (weak) consistency. They are checked in the quickest and easiest way. In addition, it is practically reasonable to consider a relaxed payoff consistency. The relaxed payoff consistency by (91)–(93) means that, as the sampling density minimally increases (in each of the three ways of the sampling increment), the game optimal value change in an appropriate approximation may grow at most by ε. The weak consistency itself is a relaxation to the consistency, where the minimal decrement of the sampling density is ignored. Therefore, the suggested method of finite approximation of staircase zero-sum games consists in the independent samplings, solving "smaller" matrix games, and stacking their solutions if they are consistent. The finite approximation is regarded appropriate if at least the respective approximate (stacked) solution is ε-payoff consistent (Definition 10).

One can notice that, in staircase game (8) decomposed into games (19), the payoff value depends only on the subinterval length if time t is not explicitly included into the function under the integral in (6). If the subinterval length does not change, every subinterval has the same matrix game. The triviality of the equal-length-subinterval solution is explained by a standstill of the players' strategies. Time variable t explicitly included into (6) means that the players may develop their actions due to the game-modelled system changes (develops) as time goes by.

Finite approximation of games played in staircase-function continuous spaces will be extended and advanced also for the case of non-antagonistic interests of two players sampling their strategy value sets irregularly. An approach to solving the corresponding "smaller" bimatrix games is not straightforwardly deduced from Theorem 2 as the optimality in the matrix game does not have an analogy for the bimatrix game [1, 6, 12, 14, 15]. The independence of the player's sampling step selection may have a deeper incompatibility impact in the bimatrix game case, where multiple and non-equivalent solutions are very often possible, which requires additional reconciliation of the varying profitability.

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В. В. Романюк

СКІНЧЕННА АПРОКСИМАЦІЯ ІГОР З НУЛЬОВОЮ СУМОЮ, ЩО РОЗІГРУЮТЬСЯ У НЕПЕРЕРВНИХ ПРОСТОРАХ СХОДИНКОВИХ ФУНКЦІЙ

Проблематика. Існує відомий спосіб апроксимації неперервних ігор з нульовою сумою, де наближений розв'язок вважається прийнятним, якщо він змінюється мінімально за мінімальної зміни кроку дискретизації. Однак цей метод не можна прямо застосувати до гри з нульовою сумою, що розігрується зі стратегіями у формі сходинкових функцій. Крім того, слід брати до уваги незалежність вибору гравцем кроку дискретизації.

Мета дослідження. Мета полягає у тому, щоб розробити метод скінченної апроксимації ігор з нульовою сумою, які розігруються у неперервних просторах сходинкових функцій, беручи до уваги, що гравці, ймовірно, дискретизують множини своїх чистих стратегій самостійно.

Методика реалізації. Для досягнення зазначеної мети формалізується гра з нульовою сумою, в якій стратегії гравців є сходинковими функціями часу. У такій грі множина чистих стратегій гравця є континуумом сходинкових функцій часу, і час вважається дискретним. Умови дискретизації множини можливих значень чистої стратегії гравця викладаються так, що гра стає визначеною на добутку скінченних просторів сходинкових функцій. Загалом, крок дискретизації у кожного гравця різний, і розподіл вибіркових точок (значень функції-стратегії) неоднорідний.

Результати дослідження. Представлено метод скінченної апроксимації ігор з нульовою сумою, які розігруються у неперервних просторах сходинкових функцій. Метод полягає у нерегулярній дискретизації множини значень чистої стратегії гравця, розв'язуванні матричних ігор меншого розміру, кожна з яких визначена на підінтервалі, де значення чистої стратегії є постійним, й укладанні їхніх розв'язків, якщо вони є узгодженими. Уклад розв'язків матричних ігор меншого розміру є наближеним розв'язком вихідної сходинкової гри. Досліджується (слабка) узгодженість наближеного розв'язку тим, наскільки змінюється виграш та оптимальна ситуація, коли щільність дискретизації мінімально збільшується трьома способами: лише приріст у першого гравця, лише приріст у другого гравця, приріст в обох гравців. Узгодженість розкладається на узгодженість виграшів, узгодженість потужності спектру оптимальної стратегії, узгодженість щільності дискретизації оптимальної стратегії та узгодженість спектральних імовірностей. З практичної точки зору доцільно розглядати релаксовану узгодженість виграшів.

Висновки. Запропонований метод скінченної апроксимації сходинкових ігор з нульовою сумою полягає у незалежних дискретизаціях, розв'язуванні матричних ігор меншого розміру за прийнятний проміжок часу та укладенні їхніх розв'язків, якщо вони є узгодженими. Скінченне наближення вважається прийнятним, якщо принаймні відповідний наближений (укладений) розв'язок є узгодженим за ε-виграшами.

Ключові слова: теорія ігор; функціонал виграшів; стратегія у формі сходинкової функції; матрична гра; нерегулярна дискретизація; узгодженість наближеного розв'язку.

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